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ABSTRACT

Designed to accompany a series of videotapes, this textbook provides information, examples, problems, and solutions relating to mathematics and its applications in technical fields. Chapter I deals with basic arithmetic, providing information on fractions, decimals, ratios, proportions, percentages, and order of operations. Chapter II focuses on algebra, covering real numbers; terms, variables, and expressions; exponents; polynomials; multiplying two expressions; factoring; algebraic fractions; signed numbers; the rectangular coordinate system; first degree, second degree, and linear equations; inequalities; and functions. After chapter III presents the geometric concepts of area, pi, volume, and conic sections, chapter IV provides an introduction to trigonometry, covering positive, negative, general, acute, and quadrantal angles; vectors; right and oblique triangles; graphing of trigonometric functions; calculators with trig functions; period of waves; and trigonometric curves. Next, chapter V focuses on differentiation and integration in calculus. Finally, chapter VI considers the applications of mathematics in plane surveying, computer science, machine parts inspection, automotive technology, aviation, manufacture engineering processes, pharmacology, and optical technology. Practice sheets with answers are included. (AYC)

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TECHNICAL MATHEMATICS

RESTRUCTURE OF TECHNICAL MATHEMATICS

Carol A. Flannery

JC 830 147

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BASIC ARITHMETIC

Chapter I

1. Fractions, Decimals
2. Ratio, Proportions
3. Percent
4. Order of Operations

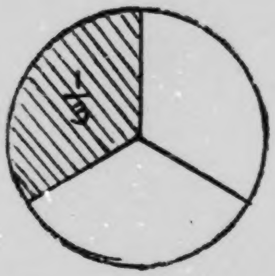
FRACTIONS

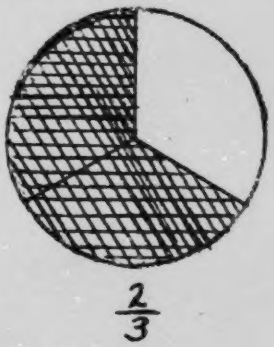
Basic Definitions

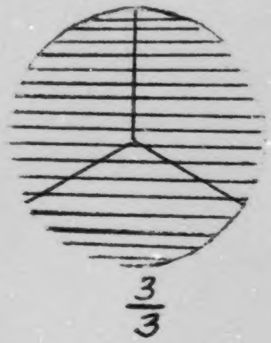
There are two basic ways of defining a fraction, such as 2/3. The first **method** is through the use of division:

$$2/3 \text{ implies } 2 \div 3, \text{ or } \frac{2}{3}$$

A second method would be to consider an object, for instance a circle, which is divided into 3 equal parts. If you had one part of the circle you would have 1/3, two parts, 2/3. And, having three parts of the circle, 3/3, would yield 1, the whole circle.







The bottom number of a fraction is called the denominator, which means "to name". The top number is called the numerator, which means "number".

Fractions can take three different forms. Proper, improper, and mixed. A proper fraction has a numerator less than the denominator. An improper fraction has a denominator less than or equal to the numerator. A mixed fraction consists of a whole number plus a fraction.

Examples:

Proper → 2/3, 3/4, 5/8

Improper → 7/3, 5/4, 9/7

Mixed → $1\frac{2}{3}$, $41\frac{37}{38}$, $1\frac{5}{7}$

It is important to note that a mixed fraction, such as $1\frac{2}{3}$, implies addition.

$$1\frac{2}{3} \rightarrow 1 + \frac{2}{3}$$

Changing Improper Fractions to Mixed Fractions

To change an improper fraction into a mixed fraction, simply divide the numerator by the denominator. If there is a remainder left over, write it divided by the denominator.

Example:

$$\frac{29}{7} = \begin{array}{r} 4\ \text{r}1 \\ 7\,\overline{)\,29} \\ \underline{28} \\ 1 \end{array} = 4\frac{1}{7}$$

$$\frac{25}{5} = \begin{array}{r} 5 \\ 5\,\overline{)\,25} \\ \underline{25} \\ 0 \end{array} = 5$$

Changing Mixed Fractions to Improper Fractions

To change a mixed fraction into an improper fraction, multiply the whole number part times the denominator. Next, add to the numerator. Write the result of this divided by the denominator.

Example:

$$5\frac{2}{3} = \frac{(3 \cdot 5) + 2}{3} = \frac{15 + 2}{3} = \frac{17}{3}$$

$$7\frac{1}{4} = \frac{(4 \cdot 7) + 1}{7} = \frac{28 + 1}{7} = \frac{29}{7}$$

Reduce to Lowest Terms

To reduce a fraction, simply find a positive integer, other than 1, which divides evenly into both denominator and numerator. When this is completed, a fraction is said to be reduced, or an equivalent fraction in lowest terms.

Example:

$$\frac{6}{8} = \frac{6 \div 2}{8 \div 2} = \frac{3}{4}$$

$$\frac{24}{36} = \frac{24 \div 6}{36 \div 6} = \frac{4}{6} = \frac{4 \div 2}{6 \div 2} = \frac{2}{3}$$

It is important to note that the fraction $\frac{6}{8}$ is just another way of expressing the fraction $\frac{3}{4}$ i.e. $\frac{6}{8} = \frac{3}{4}$. Therefore, $\frac{6}{8}$ and $\frac{3}{4}$ are "equivalent fractions".

Increasing a Fraction to Higher Terms

When solving problems, it is sometimes necessary to increase a fraction to an equivalent fraction in higher terms. Therefore, find a positive integer, other than 1 and multiply both numerator and denominator.

Example:

$$\frac{1}{3} = \frac{1 \cdot 2}{3 \cdot 2} = \frac{2}{6}$$

$$\frac{2}{6} = \frac{2 \cdot 3}{6 \cdot 3} = \frac{6}{18}$$

$$\frac{6}{18} = \frac{6 \cdot 4}{18 \cdot 4} = \frac{24}{72}$$, and so on

Adding and Subtracting Fractions

In order to add or subtract two fractions, their denominators must be alike, or equal. To make them equal, but not change the value of the fractions, we must find the smallest number in which the denominators will both divide into evenly. This number is called the Least Common Multiple, denoted LCM. One method is to start doubling the larger of the two numbers until you find a value which is evenly divisable by the smaller.

Example:

Find the LCM of 3 and 4.

4 x 1 = 4 Not divisable by 3

4 x 2 = 8 Not divisable by 3

4 x 3 = 12 Divisable by 3, therefore, 12 is LCM

After the LCM is found, we need to express the denominator in terms of the LCM. This is done through multiplication by a fraction equal to one.

Example:

Express $\frac{2}{3}$ and $\frac{1}{8}$ in terms of 24ths.

$$\frac{2}{3} = \frac{2}{3} \cdot 1 = \frac{2}{3} \cdot \frac{8}{8} = \frac{16}{24}$$

$$\frac{1}{8} = \frac{1}{8} \cdot 1 = \frac{1}{8} \cdot \frac{3}{3} = \frac{3}{24}$$

Once we have a common denominator, (a denominator shared by both fractions) addition and subtraction is accomplished by adding or subtracting the numerators and carrying over the denominator.

Solve the following problems:

1) $\frac{1}{8} + \frac{2}{3}$

Solution:

The LCM of 8 and 3 is 24. (Because, both 8 and 3 will divide evenly into 24).

$$\frac{1}{8} + \frac{2}{3} = \frac{1}{8}\left(\frac{3}{3}\right) + \frac{2}{3}\left(\frac{8}{8}\right)$$

$$= \frac{3}{24} + \frac{16}{24}$$

$$= \frac{3 + 16}{24}$$

$$= \frac{19}{24}$$

2) $-\frac{3}{7} - \frac{1}{15}$

Solution:

The LCM of 7 and 15 is 105.

$$-\frac{3}{7} - \frac{1}{15} = \frac{3}{7} \bullet \frac{15}{15} - \frac{1}{15} \bullet \frac{7}{7}$$

$$= \frac{45}{105} - \frac{7}{105}$$

$$= \frac{38}{105}$$

3) $2\,\frac{3}{4} - 1\,\frac{5}{6}$

Solution:

The LCM of 4 and 6 is 12.

$$2\,\frac{3}{4} - 1\,\frac{5}{6} = \frac{8 + 3}{4} - \frac{6 + 5}{6}$$

$$= \frac{11}{4} - \frac{11}{6}$$

$$= \frac{33}{12} - \frac{22}{12}$$

$$= \frac{11}{12}$$

Multiplying and Dividing Fractions

In order to multiply two fractions, find the product of the numerators divided by the product of the denominators and then reduce the resulting fraction:

Example:

$$\frac{2}{3} \cdot \frac{4}{8} = \frac{2 \cdot 4}{3 \cdot 8} = \frac{8}{24}$$

If one or more of the fractions are in mixed form, change the mixed fraction into improper form and multiply.

$$2\,\frac{2}{3} \cdot 4\,\frac{1}{8} = \frac{(3 \cdot 2) + 2}{3} \cdot \frac{(8 \cdot 4) + 1}{8}$$

$$= \frac{8}{3} \cdot \frac{33}{8}$$

$$= \frac{264}{24}$$

In order to divide two fractions, invert the divisor and multiply.

Example:

$$1) \quad \frac{2}{3} \div \frac{4}{8} = \frac{2}{3} \cdot \frac{8}{4}$$

$$= \frac{16}{12}$$

$$= \frac{4}{3} \quad \text{(reduced)}$$

$$= 1\,\frac{1}{3} \quad \text{(mixed form)}$$

2) $$\begin{aligned} 3\tfrac{3}{5} \div 2\tfrac{1}{3} &= \frac{18}{5} \div \frac{7}{3} \\ &= \frac{18}{5} \cdot \frac{3}{7} \\ &= \frac{54}{35} \\ &= 1\tfrac{19}{35} \end{aligned}$$

Remember:

$$\begin{array}{r} \text{QUOTIENT} \\ \text{DIVISOR}\,\overline{\big)\,\text{DIVIDEND}} \end{array}$$

In other words:

Fraction ÷ Fraction = Quotient

↓ ↓

Dividend ÷ Divisor = Quotient

Multiply or Divide as Indicated:

1) $\frac{3}{4} \cdot \frac{8}{3}$

2) $1\frac{2}{3} \cdot \frac{4}{5}$

3) $-\frac{9}{15} \div \frac{3}{5}$

4) $1\frac{2}{3} \div \frac{2}{3}$

5) $2\frac{3}{4} \cdot 3\frac{1}{8}$

6) $-2\frac{1}{2} \cdot -3\frac{2}{3}$

7) $\frac{5}{8} \div (-\frac{1}{8})$

8) $\dfrac{\frac{5}{16}}{\frac{3}{8}}$

9) $\frac{2}{3} \cdot -\frac{5}{13} \cdot \frac{39}{2}$

10) $\dfrac{\frac{5}{6} \div \frac{5}{9}}{\frac{3}{2}}$

DECIMALS

It is necessary to know how to write a fraction in decimal form and how to write a decimal in fractional terms.

Decimal Fractions

Fractions having a denominator that can be expressed as a "power of 10", are called decimal fractions. The illustration states:

Fraction	Denominator as a Power of Ten	Decimal Place
$\frac{2}{10}$	$\frac{2}{10^1}$	$0\,.\,\underline{2}\;\underline{0}\;\underline{0}\;\underline{0}$
$\frac{7}{100}$	$\frac{7}{10^2}$	$0\,.\,\underline{0}\;\underline{7}\;\underline{0}\;\underline{0}$
$\frac{23}{1000}$	$\frac{23}{10^3}$	$0\,.\,\underline{0}\;\underline{2}\;\underline{3}\;\underline{0}$
$\frac{8}{10,000}$	$\frac{8}{10^4}$	$0\,.\,\underline{0}\;\underline{0}\;\underline{0}\;\underline{8}$

The power of ten indicates the decimal place of the last digit in the numerator. For example:

$$\begin{array}{cccccccl} 0 & . & \_ & \_ & \_ & \_ & & \text{and so on . . . .} \\ \downarrow & & \downarrow & \downarrow & \downarrow & \downarrow & & \\ \text{Whole Number Part} & . & 10^1 & 10^2 & 10^3 & 10^4 & & \end{array}$$

In the fraction $\frac{24}{1000}$, the numerator consists of two digits: 2 and 4. The power of ten in the denominator is 3, because $1{,}000 = 10^3$. Therefore, the last digit of the numerator, 4, must be in the decimal place representing 10^3.

$$\frac{24}{1000} = 0\ .\ \underline{0}\ \underline{2}\ \underline{4}\ \underline{\ \ }$$

As another example, consider:

$$\frac{3}{10{,}000} = 0\ .\ \underline{0}\ \underline{0}\ \underline{0}\ \underline{3}$$

The last digit in the numerator is the only digit: 3. The denominator 10,000 can be expressed as 10^4. Therefore, the digit 3 is placed in the decimal slot representing 10^4. The other slots preceding 3, would have zeros.

Negative integer exponents of 10 are expressed as:

$$10^{-1} = \frac{1}{10^1} = \frac{1}{10}$$

$$10^{-2} = \frac{1}{10^2} = \frac{1}{100}$$

$$10^{-3} = \frac{1}{10^3} = \frac{1}{1{,}000}$$

This follows the definition of negative exponents.

Changing Fractions to Decimals

The definition of a fraction concerns division. To change a fraction into decimal form, simply divide the denominator into the numerator. Since the denominator is usually larger than the numerator, we must place a decimal point after the numerator and add zeros in order to carry out division. For example:

FRACTION → DECIMAL

Fraction	Procedure	Result
$\frac{3}{8}$	$\begin{array}{r} .375 \\ 8/\overline{3.0000} \\ \underline{2\ 4} \\ 60 \\ \underline{56} \\ 40 \\ \underline{40} \\ 0 \end{array}$ Stop dividing when zero is the remainder.	0.375

Changing Decimals to Fractions

Decimals have a denominator depending upon the place of the last digit of the numerator. Observe:

DECIMAL → FRACTION

Decimal	Observation	Fraction
0.3127	The last digit is in the 10^4 place.	$\frac{3127}{10^4}$
3.76	Decimal consists of a whole number and decimal part.	$3 + \frac{76}{100} = 3\frac{76}{100}$

Convert to Decimal Form

1) 3/4
2) 2 5/8
3) 4 1/2
4) 2/3
5) 3 1/64
6) 3/32
7) 4 3/8
8) 5/4
9) 1/4
10) 11/16

Convert to Fractional Form

1) 1.3750
2) 2.50
3) .667
4) .875
5) .025
6) 2.0007
7) 4.05
8) -3.15
9) .0625
10) 3.75

RATIO AND PROPORTION

Ratio

When we speak of fractions, we are actually expressing ratios. However, the concept of ratios is also to establish a relationship between the numerator and denominator. This relationship helps to solve problems in many sciences.

To say, "55 miles per hour", is expressing a ratio: The number of miles that can be travelled in an hour of time.

Ratio Notation: $\frac{\text{55 miles}}{\text{hour}} \rightarrow$ 55 miles per hour

From basic formulas, we can obtain ratios. Another term used when dealing with ratios is, conversion factors. The formulas listed help to explain.

Area of a Rectangle = $(l)(w)$

Due to manipulation of this formula, using algebra techniques, the formula can be expressed as:

Conversion #1: $A = (l)(w)$

Conversion #2: $\frac{A}{w} = l$

Conversion #3: $\frac{A}{l} = w$

Conversion #1 is the basic formula to find area when length and width are known. From the basic formula, dividing both sides by w, the formula is now expressed as Conversion #2. Dividing both sides by l, the formula is now in the form of "length equals . . .", shown in Conversion #3.

Conversion factors exist also in linear measurement. For instance:

$$\boxed{\frac{12\ \text{inches}}{1\ \text{foot}}} \qquad \boxed{\frac{3\ \text{feet}}{1\ \text{yard}}} \qquad \boxed{\frac{5{,}280\ \text{ft}}{1\ \text{mile}}}$$

It is also correct to write the factors as:

$$\boxed{\frac{1\ \text{foot}}{12\ \text{inches}}} \qquad \boxed{\frac{1\ \text{yard}}{3\ \text{feet}}} \qquad \boxed{\frac{1\ \text{mile}}{5{,}280\ \text{ft}}}$$

As long as the quantity relationship is correct, i.e., 12 inches = 1 foot, etc., it doesn't matter how the ratio (conversion factor) is expressed.

The reason we can write a quantity ratio two different ways, is dependent upon the problems to be solved. For example, suppose it was necessary to find out how many feet are in 2 miles:

$$\frac{2\ \text{miles}}{1} \cdot \left(\frac{5{,}280\ \text{ft}}{1\ \text{mile}}\right) = 10{,}560\ \text{feet}$$

The calculation is done by:

$$\frac{\text{QUANTITY TO BE CHANGED}}{1} \cdot (\text{CONVERSION FACTOR}) = \text{RESULT}$$

Always place the quantity to be changed over the whole number 1. We used the conversion factor $\frac{5{,}280\ \text{feet}}{1\ \text{mile}}$ because with the unit "mile" in the denominator, the units "mile" cancel out, leaving our desired units "feet".

To find out how many inches are in 2 miles, we use two conversion factors:

$$\frac{2\ \cancel{\text{miles}}}{1}\cdot\left(\frac{5{,}280\ \cancel{\text{ft}}}{1\ \cancel{\text{mile}}}\right)\cdot\left(\frac{12\ \text{inches}}{1\ \cancel{\text{foot}}}\right) = \frac{(2)(5{,}280)(12\ \text{inches})}{(1)\ (1)\ (1)}$$

$$= 126{,}720\ \text{inches}$$

Another example, could involve the following ratios related to liquid measurement:

$$\boxed{\frac{4\ \text{quarts}}{1\ \text{gallon}}}\quad\text{or}\quad\boxed{\frac{1\ \text{gallon}}{4\ \text{quarts}}}$$

$$\boxed{\frac{2\ \text{pints}}{1\ \text{quart}}}\quad\text{or}\quad\boxed{\frac{1\ \text{quart}}{2\ \text{pints}}}$$

$$\boxed{\frac{16\ \text{ounces}}{1\ \text{pint}}}\quad\text{or}\quad\boxed{\frac{1\ \text{pint}}{16\ \text{ounces}}}$$

Using the listed conversion factors, calculate the number of ounces in 3.5 gallons:

$$\frac{3.5\ \cancel{\text{gallons}}}{1}\cdot\left(\frac{4\ \cancel{\text{quarts}}}{1\ \cancel{\text{gallon}}}\right)\cdot\left(\frac{2\ \cancel{\text{pints}}}{1\ \cancel{\text{quart}}}\right)\cdot\left(\frac{16\ \text{ounces}}{1\ \cancel{\text{pint}}}\right)$$

Solving:

$$\frac{3.5}{1}\cdot\left(\frac{4}{1}\right)\cdot\left(\frac{2}{1}\right)\cdot\left(\frac{16\ \text{ounces}}{1}\right) = 448\ \text{ounces}$$

Proportion

By definition, a proportion is the equality of two ratios. For example, if Car #1 has a maximum speed of 150 mph, and Car #2 has a maximum speed of 195 mph, the ratio of their speeds is a proportion:

$$\frac{\text{Car \#1 Speed}}{\text{Car \#2 Speed}} \begin{array}{c}\rightarrow\\ \rightarrow\end{array} \frac{150}{195} = \frac{10}{13}$$

The following proportions have one unknown quantity. Solving for the unknown, the equality can be seen:

EXAMPLE

$$\frac{2}{3} \times \frac{6}{x}$$

$$2x = 18 \qquad \text{means} = \text{extremes}$$

$$x = 9$$

EXAMPLE

$$\frac{8}{12} \times \frac{6}{x}$$

$$8x = 72 \qquad \text{means} = \text{extremes}$$

$$x = 9$$

EXAMPLE

If a machinist can produce 12 machine parts in 4 hours, how many parts can be produced in a 40 hour work week?

$$\frac{12 \text{ parts}}{4 \text{ hours}} = \frac{x}{40 \text{ hours}}$$

$$(12 \text{ parts})(40 \text{ hours}) = (x)(4 \text{ hours})$$

$$\frac{(12 \text{ parts})(40 \text{ h}\cancel{\text{ours}})}{(4 \text{ h}\cancel{\text{ours}})} = x$$

$$12 \text{ parts} = x$$

EXAMPLE

If a current produces 20 calories per second in a resistance of 2 ohms, how much heat is produced in a resistance of 10 ohms?

$$\frac{20 \text{ calories}}{2 \text{ ohms}} = \frac{x}{10 \text{ ohms}}$$

$$(20 \text{ calories})(10 \text{ ohms}) = (x)(2 \text{ ohms})$$

$$\frac{(20 \text{ calories})(10 \cancel{\text{ohms}})}{(2 \text{ o}\cancel{\text{hms}})} = x$$

$$100 \text{ calories} = x$$

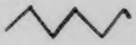
Applications in Electricity

Proportions and equations involving more than two unknowns occur in the study of electricity as well as many other sciences. One of the first "laws" introduced to an electronics student, is Ohm's Law.

$$\text{ELECTRICAL RESISTANCE, in ohms} = \frac{\text{POTENTIAL DIFFERENCE, in volts}}{\text{CURRENT, in amperes}}$$

Written symbolically, and further explained:

$$R = \frac{V}{I} \begin{matrix} \rightarrow \\ \rightarrow \end{matrix} \frac{\text{Work per unit positive charge from one body to another.}}{\text{Quantity of charge per unit of time.}}$$

In diagrams illustrating circuits, a saw-toothed line () depicts resistance. The unit of ohms, is represented by Ω, from the Greek letter, omega.

By conversion factors, from Ohm's Law we can derive the equations for potential difference V, and current, I:

$R = \frac{V}{I}$ → Ohm's Law

$IR = V$ Obtained by multiplying both sides of Ohm's Law by a factor of $\frac{I}{I}$.

$\frac{V}{R} = I$ Obtained by dividing both sides of the second **equation**, by R.

With a little practice, you can manipulate given equations, solving for other unknown quantities, in terms of unknowns.

Problems:

1) If a circuit has a resistance of 30 ohms, and a current of 10 amps, calculate the potential difference.

2) A dry cell is connected to a small light bulb. If the potential difference across the bulb is 2 volts, and the current is 0.5 amps, what is the resistance of the circuit?

3) Suppose a curcuit maintained a resistance of 3 ohms, with a potential difference of 1.5 V. What is the current flowing through the circuit?

Measurements of Electricity in Conversion Form

I. A unit "quantity of electricity" is the coulomb.

$$\boxed{\frac{1\ \text{coulomb}}{6.24 \times 10^{18}\ \text{electrons}}}$$

II. The unit of "electric current" is called the ampere.

$$\boxed{\frac{1\ \text{ampere}}{1\ \text{coulomb/sec}}}$$

III. A watt is the unit of "electrical power". The power in watts corresponds to: (voltage of a direct current circuit) x (current).

$$\boxed{\begin{array}{c}\text{Power, in watts} = (\text{Voltage})(\text{Current}),\ \text{or} \\ P = VI\end{array}}$$

Other measurements:

$$\boxed{\frac{1\ \text{Kilowatt}}{1000\ \text{watts}}} \qquad \boxed{\frac{1\ \text{kilowatt}}{1.341\ \text{horse power}}}$$

Surveyor's Chain Measure as Conversion Factors

$\boxed{\dfrac{7.92 \text{ in}}{1 \text{ link}}}$ $\boxed{\dfrac{100 \text{ links}}{1 \text{ chain}}}$ $\boxed{\dfrac{80 \text{ chains}}{1 \text{ statute mile}}}$

EXAMPLE:

How many feet are in 100 links?

$$\frac{100 \text{ links}}{1} \cdot \left(\frac{7.92 \text{ in}}{1 \text{ link}}\right) \cdot \left(\frac{1 \text{ foot}}{12 \text{ in}}\right) =$$

Problems:

1. How many links are in 3.5 feet?

2. Determine the number of chains in 8,500 inches.

3. Find out the number of links in 3 miles.

4. How many inches are in 342 links?

RATIO AND PROPORTION

I. Problem:

If a person earns \$250 per week for how many weeks must she work to earn \$12,500?

Solution:

If X is the unknown number of weeks then \$250 is to \$12,500 as 1 week is to X weeks, i.e.

$$\frac{250}{12500} = \frac{1}{X}$$

or

$$X = 12500/250 = 50 \text{ weeks.}$$

II. Problem:

A vehicle has an axle ratio of 3.5:1, i.e., the drive shaft makes 3.5 revolutions for each revolution of the axle. If the drive shaft is turning at 875 rpm what is the rpm of the axle?

Solution:

The ratio of the speeds is the same as the axle ratio. Therefore

$$\frac{3.5}{1} = \frac{875}{X}$$ where X is the speed of the axle.

Solving for X: $X = 875/3.5 = 250$ rpm.

III. Problem:

Suppose the headlights of an automobile can point down 1 inch in every 12 feet. How far down the road will the lights be pointed if they are 24 inches above the ground?

Solution:

Let the distance down the road be X. Then

$$\frac{1}{12} = \frac{24}{X}$$

(corresponding sides of similar triangles.)

So $$X = 24 \cdot 12 = 288 \text{ feet.}$$

CONVERSION OF UNITS

I. Problem:

An oil change is recommended **after** 7,500 miles for a certain engine. For how many kilometers can it be driven before an oil change is desirable?

Solution:

The 'multiply-by-one' method is a convenient system for converting from one system of units to another.

Since: 1 mile = 1.6 kilometers

then:

$$\frac{1.6 \text{ km}}{1 \text{ mi}} = 1.$$

The value 7,500 mi can be multiplied by $\frac{1.6 \text{ km}}{1 \text{ mi}} = 1$ to change the units without changing the value of the distance; i.e.

$$7{,}500 \cancel{\text{mi}} \cdot \frac{1.6 \text{ km}}{1 \cancel{\text{mi}}} = 12{,}000 \text{ km}.$$

II. Problem:

A truck is rated as 2,744 kilograms gross vehicle weight rating (GVWR). What is its GVWR in pounds?

Solution:

The relation between pounds and kilograms is:

$$1 \text{ kg} = 2.2 \text{ lbs}$$

or $$\frac{2.2 \text{ lbs}}{1 \text{ kg}} = 1$$

Multiplying: $2744 \cancel{\text{kg}} \cdot \frac{2.2 \text{ lbs}}{1 \cancel{\text{kg}}} = 6036.8 \text{ lbs}$

1) A man drives 30 miles in 3.5 hours.

   a. How far will he go in 2 hours?

   b. How long will it take him to drive 75 miles?

2) The price of beef is $3.00 per pound.

   a. How many pounds will $4.50 buy?

   b. How much will 7.3 pounds cost?

3) If y varies with x, and if y = 20 when x = 4

   a. Find y when x = 6.

   b. Find x when y = 25.

4) An automobiles gas mileage is 20 miles per gallon of gasoline.

   a. How many miles will the car go on 15 gallons?

   b. How many gallons will it take to drive 440 miles?

5) In a simple circuit, the voltage, in volts, is proportional to the current in amps. If e = 12 volts when i = 6 amps, find e when i = 8 amps.

Solve for the Missing Term:

1) $\frac{1}{2} = \frac{3}{x}$

2) $\frac{3}{4} = \frac{x}{16}$

3) $\frac{1.2}{3} = \frac{y}{4}$

4) $\frac{1.5}{4} = \frac{A}{2}$

5) $\frac{A}{6} = \frac{12}{36}$

6) $\frac{4.3}{1} = \frac{8.2}{9.3}$

7) $1\frac{4}{3} = \frac{1}{x}$

8) $\frac{5}{3} = \frac{4}{x}$

9) $\frac{2.8}{1.4} = \frac{2x}{3.2}$

10) $\frac{2.5}{25} \quad \frac{x}{100}$

PERCENT

The word percent, denoted by the symbol %, came from the Latin word per, meaning "by the" and centum, meaning "hundred". Thus placing these two words together, we would have "by the hundred". Therefore, when we say 50% or 75%, we are implying that in fractional form the denominator of the number is 100 while the numerator is 50 or 75, respectfully.

$$10\% \text{ implies } \frac{10}{100} \text{ or } \frac{1}{10} \text{ or } .1$$

$$50\% \rightarrow \frac{50}{100} \text{ or } \frac{1}{2} \text{ or } .5$$

$$\tfrac{2}{3}\% \rightarrow \frac{\frac{2}{3}}{100} \text{ or } \frac{2}{300} \text{ or } .0067$$

$$x\% \rightarrow \frac{x}{100} \text{ or } (.01)x$$

It is important to note that before you attempt a percent problem, always change the percentage to its decimal or fractional equivalent.

When given a fraction, such as 3/4, change to its percent equivalent, simply by dividing the numerator by the denominator and then multiply the result by 100%. And for decimal values, multiply times 100%.

$$3/4 = .75 \times 100\% = 75\%$$

$$.283 = .283 \times 100\% = 28.3\%$$

$$\frac{A}{100} = \frac{A}{100} \times 100\% = A\%$$

In order to solve stated percent problems, it is often easier to memorize the implied meanings of a few key words rather than memorizing how to work each type of percent problem.

Key Word	Meaning	Symbol
of	times	•
is	equals	=
sum	add	+
difference	subtraction	-
what value what number	unknown value	any symbol (A - Z, ∝ , β, etc.)

Solve the following stated problems:

1) 15% of 80 is what number?

$$15\% = \frac{15}{100}$$

$$\left(\frac{15}{100}\right) \cdot 80 = x$$

$$\frac{120}{10} = x$$

$$12 = x$$

2) 10% of what number is 25?

$$\frac{10}{100} \cdot x = 25$$

$$\left(\frac{10}{100}\right) \cdot \frac{100}{10} \cdot x = 25 \cdot \frac{100}{10}$$

$$x = 250$$

3) What percent of 120 is 84?

$$\frac{x}{100} \cdot 120 = 84$$

$$x \cdot \frac{120}{100} = 84$$

$$x \cdot \frac{12}{10} \cdot \left(\frac{10}{12}\right) = 84 \cdot \left(\frac{10}{12}\right)$$

$$x = 70\%$$

4) A man purchased a car for $7,000.00 which was 22.5% of his yearly income. What is his yearly income?

$$\$7{,}000 = \frac{22.5}{100} \cdot \text{Y.I.}$$

$$\$7{,}000 = \frac{22.5\ \text{Y.I.}}{100}$$

$$\left(\frac{100}{1}\right) \cdot \$7{,}000 = \frac{22.5\ \text{Y.I.}}{100} \cdot \left(\frac{100}{1}\right)$$

$$\$700{,}000 = 22.5\ \text{Y.I.}$$

$$\frac{\$700{,}000}{22.5} = \frac{22.5\ \text{Y.I.}}{22.5}$$

$$\$31{,}111.11 \doteq \text{Y.I.}$$

Percent

Convert the following to its percent equivalent:

1) 3/4

2) 10

3) .082

4) 2/3

5. 1.05

Convert the following to its decimal equivalent:

1) 50%

2) 33 1/3%

3) .012%

4) 2.5%

5) 100%

INCREASE OR DECREASE WITHIN PERCENT

Often a value of an item is increased or decreased by some given percent of the original item. These problems are solved by simply taking the original item and adding or subtracting a percentage of that item producing its new amount:

$$\text{Item} \pm \frac{\text{Percent}}{100} \cdot (\text{Item}) = \text{New Item}$$

Solve the following problems:

1) The population of a town was 3000. During the next decade it was found to have an increase in population of 16%. What is the new population?

$$3000 + \frac{16}{100} \cdot (3000) = \text{New Population}$$

$$3480 = \text{New Population}$$

2) The price of chicken is 80 cents per pound. This week it is on sale at a savings of 20%. What is the sale price?

$$80 - \frac{20}{100} \cdot (80) = \text{Sale Price}$$

$$80 - \frac{1600}{100} = \text{Sale Price}$$

$$80 - 16 = \text{Sale Price}$$

$$64 \text{ cents per pound} = \text{Sale Price}$$

3) After an increase of 12%, the price of an item is \$156. What was the original price?

$$(\text{Item}) + \frac{\text{Percent}}{100} \cdot (\text{Item}) = \$156$$

$$x + (12\% \cdot x) = \$156$$

$$x + \left(\frac{12}{100} \cdot x\right) = \$156$$

$$x + .12x = \$156$$

$$1.12x = \$156$$

$$\frac{1.12x}{1.12} = \frac{\$156}{1.12}$$

$$x = \$139.29$$

Note:

Why is $x + .12x = 1.12x$? Remember, $x = 1x$, and can also be expressed as $1.00x$. Then, $1.00x + .12x = 1.12x$.

4) An item was selling for \$12. It is now selling for \$9. What is the percentage of difference?

$$(\text{Item}) - \frac{\text{Percent}}{100} \cdot (\text{Item}) = \text{New Item Price}$$

$$\$12 - \frac{x}{100} \cdot (12) = \$9$$

$$\$12 - \frac{12x}{100} = \$9$$

$$\begin{array}{rrl} & \$12 - \dfrac{12x}{100} & = \$9 \\ - & \underline{\$12 \qquad\quad} & \underline{-\$12} \\ & - \dfrac{12x}{100} & =-\$3 \end{array}$$

$$\left(\frac{100}{1}\right) \cdot \frac{-12x}{100} = -\$3 \cdot \left(\frac{100}{1}\right)$$

$$-12x = -\$300$$

$$\frac{-12x}{-12} = \frac{-\$300}{-12}$$

$$x = 25, \text{ or } 25\%$$

INTEREST AND PER CENT

I. Problem:

A set of tools costs $1200.00. If a downpayment of 10% is made the balance can be financed at a rate of 18% with payment due in 15 months. What will be the total amount paid for the tools?

Solution:

A) Downpayment: (0.10)($1200) = $120

B) Balance: $1200 - $120 = $1080

C) Interest: I = PRT, where P is the principal, R the rate expressed as a decimal, and T the time in years.

So I = (1080)(0.18)(1.25) =$243

D) Total: The total cost is the sum of the downpayment, balance, and interest.

$ 120
$1080
$ 243
Total $1443

II. Problem:

Suppose you earn $250 per week. If 6.9% of your salary is withheld for Social Security and 15% for Federal Income Tax what will your 'take-home' pay be?

Solution:

Social Security: (0.069)($250) = $17.25

Federal Income Tax: (0.15)($250) = $37.50

Total withheld: $17.25 + $37.50 = $54.75

Balance: $250.00 - $54.75 = $195.25

III. Problem:

An engine has cylinders with a diameter of 4 inches. If it is rebored to increase the diameter by 0.04 inches (or 1%) what will be the percentage increase in the area of the cylinders?

Solution:

The area of a circle is $\pi D^2/4$. Thus the original area is: $3.14(4^2/4) = 12.57$ sq. in.

After reboring the diameter will be 4.04 inches and the area will be: $3.14(4.04^2/4) = 12.82$ sq. in.

So the increase in area is:

$12.82 - 12.57 = 0.25$ sq. in.

The percentage increase is:

$$\frac{0.25}{12.57}(100) = 2\%$$

Solve for Missing Value:

1) 10% of what number is 12?

2) What percent of 60 is 12?

3) 60% of 55 is what value?

4) What percent of 50 is 20?

5) What number is 13% of 25?

6) A mans income is $15,000.00 per year. This year he gets a 10% raise. By what amount is his yearly income increased?

Solve for the Indicated Value:

1) After an increase of 10% the cost of a house is $55,000. What was the original cost?

2) Last week an item cost $6.00. It is now on sale for $5.00. What is the percentage of decrease?

3) The price of gas increased by 10% per gallon. It was $1.09. What price is it now?

4) A man earned $12,000.00. He now earns $13,000.00. What percent raise did he acquire?

5) The cost of living is reportedly up 15%. It was $8,000. What is it now?

ORDER OF OPERATIONS

The order of operations is important for evaluating arithmetic expressions in a consistant manner. For instance, given the expression:

$$5 \cdot 3 + 4$$

Do we first add and then multiply equaling 35 (which is incorrect), or should we multiply and then add, resulting in 19?

In order to prevent obtaining two different answers to one problem, a set of rules exist called:

The Order of Operations

I. Evaluate first all expressions within parenthesis. Always begin with inner parenthesis.

II. Next, evaluate all roots and/or powers.

III. Do all multiplication and division in order, starting from the left and going toward the right.

IV. Addition is the next procedure, with subtraction done last.

Sample Evaluations

EXAMPLE: $6 \cdot 5 \div 3 + 6 \cdot 9$

$30 \div 3 + 6 \cdot 9$

$10 + 6 \cdot 9$

$10 + 54$

64

EXAMPLE: $8\,(5 \cdot 2 + (3 - 2 \cdot 6)) - 9$

$8\,(5 \cdot 2 + -9) - 9$

$8\,(10 - 9) - 9$

$8\,(1) - 9$

$8 - 9$

-1

EXAMPLE: $2 \cdot 6^2 + 3 - 8^2 + 2 \; 9$

$$2 \cdot 36 + 3 - 64 + 2 \cdot 3$$

$$72 + 3 - 64 + 6$$

$$75 - 71$$

$$4$$

EXAMPLE: $\dfrac{6 \cdot 10 + 6}{3\,(6 + 5)}$

$$\frac{60 + 6}{3\,(11)}$$

$$\frac{66}{33}$$

$$2$$

Simplify the Following

1) $2 \cdot 3 + 5$

2) $2\,(6 + 3) + 4$

3) $6 \cdot 4 - 2 \cdot 3$

4) $8 - 5 \cdot 2 + 3$

5) $(2 + 3\,(3 \cdot 2 - 1)) + 4$

6) $-8 \cdot 2 + 3 \cdot 5$

7) $2 \cdot 3 \div 2 \cdot 4$

8) $4 \div 2 \div 2$

9) $5 \cdot 4 \div 2 + 2$

10) $(3 + 2 \cdot 3) \div (4 \cdot 2 + 1)$

ALGEBRA

Chapter II - Algebra (Part I)

1. An Introduction
2. The Real Numbers
3. Terms, Variables, and Expressions
4. Structure in Algebra

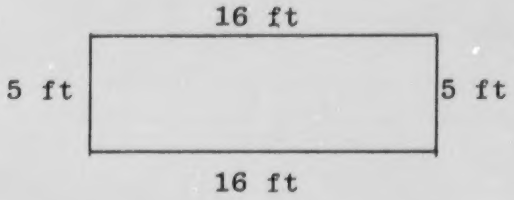
ALGEBRA

An Introduction

The solving of quadratic equations was being done 4,000 years ago in ancient Babylonia. Also, summation of limits, which touch upon the concept of integral calculus, was known in ancient times. Significant contributions to Algebra were made from not only the Babylonians, but Hindus, Arabs, and the Greeks.

Algebra is really a shorthand to arithmetic. Even though advanced algebra involves special problem solving techniques, the beginnings of algebra were based on symbols for unknowns and short-cut calculations.

For example, in arithmetic, the perimeter of a rectangle is defined as adding the lengths of all four sides:

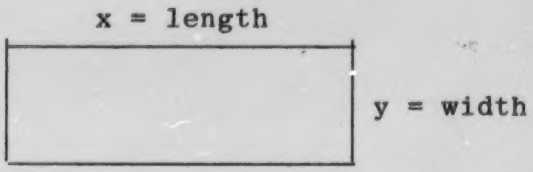


Perimeter = 16 ft + 5 ft + 16 ft + 5 ft

= 42 ft

However in algebra, symbols denote the length and width. Final calculations involve "combining" and multiplication.

Example:



$$\text{Perimeter} = 2x + 2y$$

It is interesting to note the origin of the word "algebra". Some books designate that the term is from an old Arabic word, "algebrista" -- meaning a person expertised in the ancient practice of "blood-letting" (a type of medical cure based on superstition and false assumptions concerning human diseases).

For more interesting historical data, consult *An Introduction to the History of Mathematics* - 3rd Edition, Howard Eves; Holt, Rinehart and Winston, 1969.

THE REAL NUMBERS

The single most important tool of today's science and technology is mathematics. The Real Numbers lie at the heart of the study of mathematics. To advance in mathematics, it is important to understand the Real Numbers.

What are the Real Numbers? What is their significance? One might loosely define the Real Numbers as an abstract language we can use to describe the physical characteristics of all the materials that surround us. The importance of the Real Numbers in a numerical sense is the accuracy that can be obtained in describing the physical world.

Mathematics is often referred to as the exact science. The Real Numbers, being fundamental to the study of modern mathematics, are greatly responsible for this reference. To better understand the Real Numbers we will look briefly at their history.

"Some authorities would trace the origin of mathematics back some 300,000 years to the beginning of man's cultural development."[1] Man's first great intellectual achievement in mathematics was the development of the concept of numbers. It seems reasonable that the numbers man first developed were the simple counting numbers, (e.e., 1, 2, 3, ...) which are often referred to as the Natural numbers.

"The earliest physical evidence of man's ability to count is a notched bone dating back to about 6,000 B.C."[2] Of course, the symbols used to represent the counting numbers for one civilization were not the same for a neighboring civilization. The development of the counting numbers probably arose out of a need for record keeping, construction work, land surveying and calendar computation, etc. By 2,000 B.C., China, Egypt, Mesopotamia (Babylon), and India all used some numeration system.

After counting numbers had been established, the ideas of addition and multiplication were developed. The concept of the ratio of two numbers was introduced which became what we call today, fractions, or Rational Numbers. "The Egyptians were familiar with the idea of fractions. A mathematics text from 1650 B.C., called the Ahmes (or Rhind) papyrus, shows the Egyptians were capable of using fractions and expressing parts of a whole."[3]

The Egyptians did not represent fractions as we know them today. The Egyptians representation of fractions was much more complicated than our own. "The extensive and complicated computations with fractions were one reason the Egyptians never developed arithmetic or algebra to an advanced state."[4]

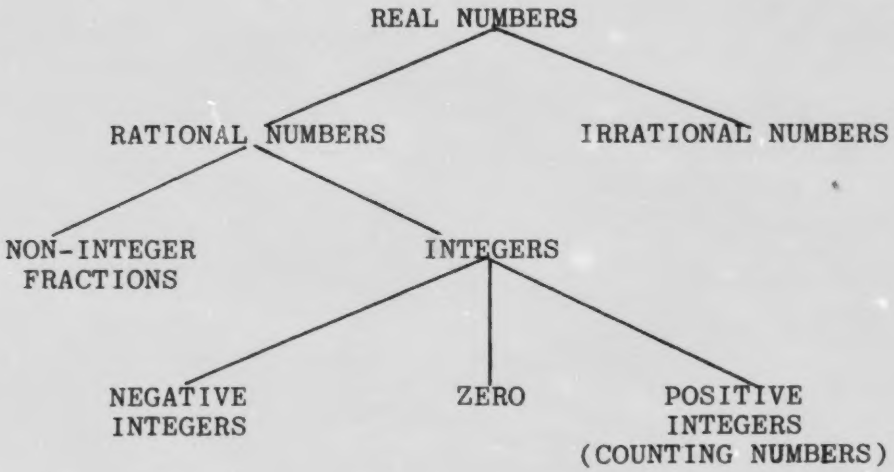
The rise of the Greek civilization brought the systematic study of mathematics. While earlier civilizations were interested in how to compute a certain problem, the Greeks wanted to know why a problem solving method worked. This led to the development of general methods for solving mathematical problems.

The Greeks soon discovered there were certain numbers that could not be represented as a fraction or counting number. This disturbed the Greeks, for it contradicted their mathematics as well as religious beliefs. However, in the 4th century, B.C., the Greek, mathematician Eudoxus accepted these 'strange' numbers and incorporated them into the Greek mathematical structure. The Greeks labeled these mysterious numbers, Irrational Numbers.

THE BASE TEN NUMERATION SYSTEM

After the Greek civilization declined from power, very little was done to advance the study of mathematics for several hundred years. About 500 A.D. the Hindus of India began to excel in mathematics. The Hindus translated many of the remaining Greek works involving mathematics and became a dominant force in the advancement of mathematics during the middle ages. The Hindus developed the base ten (decimal) numeration system. No other numeration system could match the base ten system in terms of simplicity and overall effectiveness. The base ten numeration system incorporated the idea of positional notation and introduced a new number, zero, to be used as a place holder. The Hindus introduced the base ten system to the Arabs of the middle east who quickly accepted the new system and popularized it in their writings. Today the base ten numeration system is used world-wide and is commonly referred to as the Hindu - Arabic numeration system.

The structure of the Real Numbers can be visualized by the following diagram:


REAL NUMBERS
RATIONAL NUMBERS
IRRATIONAL NUMBERS
NON-INTEGER FRACTIONS
INTEGERS
NEGATIVE INTEGERS
ZERO
POSITIVE INTEGERS (COUNTING NUMBERS)

At this point, it has been shown that the Real Number System can be broken into two subgroups:

Rational and Irrational Numbers. . . .

The Rational Numbers can be further broken down into the subgroups:

Integers and Non-Integer Fractions. . .

INTEGERS

Positive integers are the same as the counting numbers that man has used for centuries, (1, 2, 3, . . .). If we include zero, we have a group called:

WHOLE NUMBERS ≡ {0, 1, 2, 3, . . .}

Negative integers are the negative counterparts of the positive integers. It is important to note here that zero is considered neither positive or negative. The combination of the positive and negative integers with zero, comprises the group:

INTEGERS ≡ {. . . -3, -2, -1, 0, 1, 2, 3, . .}

NON-INTEGER FRACTIONS

Simply, non-integer fractions are those numbers which can be expressed in fractional form (i.e. a ratio of two integers), but cannot be expressed as an integer. For example: 1/2, 2/3, 3/4, are all non-integer fractions.

RATIONAL NUMBERS

A Rational Number is a number which can be expressed as a ratio of two integers (i.e. a fraction). Earlier, we indicated that all Rational Numbers could be classified into one of two groups, Integer or Non-Integer Fraction. It is important to understand that any Integer can be expressed as a fraction.

Rational numbers written in decimal form, have one of the two characteristics:

1) Terminate after a finite number of digits, for example:

   $1/8 = 0.125$
   ↑
   terminates here

   OR

2) Repeat a certain pattern of digits, infinitely. For example:

   $1/3 = 0.333\overline{3}$

NOTE: The bar drawn over the digit, 3, indicates that the number 3 is repeated infinitely.

IRRATIONAL NUMBERS

An irrational number cannot be expressed as a ratio of two integers. To identify an irrational number it is often necessary to investigate its decimal form. An irrational number has a non-terminating, non-repeating decimal form.

For example:

$$\sqrt{2} = 1.4142135\ldots\text{(decimal form)}$$

The three dots at the end of decimal number indicate that the sequence of digits continues on infinitely. Also note the absence of any pattern of repetition among the digits. If a decimal number exhibits any pattern of repetition that carries on infinitely throughout the sequence, then the number is not classified as irrational.

Some irrational numbers are represented by special symbols. For example, the Greek letter π

$$\pi \text{ (pronounced pi)} = 3.14159\ldots$$

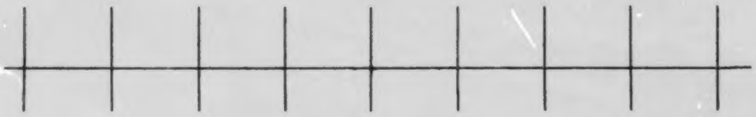
In real world applications, irrational numbers are approximated by rational numbers, the number of significant digits retained is dependent upon the accuracy required for a certain problem.

For example: $\pi \cong 3.14$

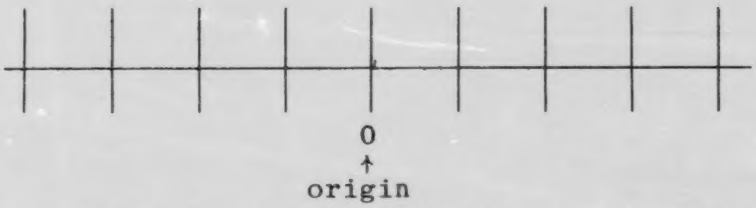
$$\sqrt{2} \cong 1.414$$

THE REAL NUMBER LINE

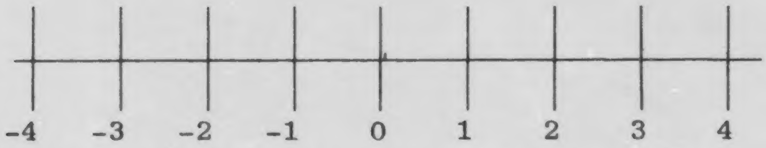
It is often convenient to visualize the Real Numbers as points on a line. We start by drawing a horizontal line and dividing it into segments of equal length.



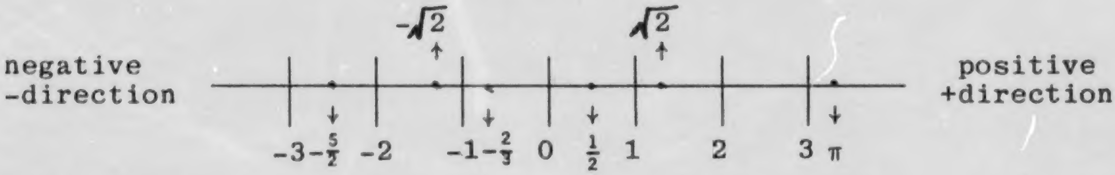
Designate some point on the line as the origin and place the Real Number, 0, at this point.



At each interval to the right of the origin, place the positive integers. And, at each interval to the left of the original place the negative integers.



The rational numbers that cannot be represented as integers, are located between the positions of the integers. Remaining points between the integers represent irrational numbers. On the following number line we have labeled a few common real numbers:


negative
-direction
$-\sqrt{2}$
$\sqrt{2}$
positive
+direction
-3
$-\frac{5}{2}$
-2
-1
$-\frac{2}{3}$
0
$\frac{1}{2}$
1
2
3
π

Note that the Real Numbers are positioned on the line such that as we proceed to the right (positive direction), the numbers become larger. Proceeding to the left (negative direction), the numbers become smaller.

We call this representation of real numbers, the Real Number Line. The Real Number Line has many uses and will be utilized in certain following sections.

FOOTNOTES

[1] Hackworth and Howland, Introductory College Mathematics - History of the Real Numbers, (Philadelphia, Pa: W. B. Saunders, Co., 1976).

[2] Ibid., pg. 3.

[3] Ibid., pg. 7.

[4] Kline, "Mathematical Thought from Ancient to Modern Times", (New York, N.Y.: Oxford University Press, 1972) pg. 18.

Terms

A term is an algebraic expression that:

1) Consists of a single variable or constant.

$$3,\ 7,\ x,\ y$$

2) Is the indicated product or quotient of variables and/or constants.

$$3x,\ xy,\ 7xy$$

3) Is the indicated product or quotient of expressions that contain variables and/or constants.

$$(3x+y) \cdot (6x-9),\ 3x \cdot (6y+9),\ \frac{6x+3}{4y}$$

Terms are made up of two basic parts:

1) The coefficient or real number part.
2) The variable or letter part.

Given the following terms:

$$6xy,\ y,\ 2(x+y).$$

The coefficient parts are 6, 1, and 2 respectively, while the variable parts are xy, y, $(x+y)$.

Two separate terms are considered alike when their variable parts are exactly the same. Here are some like terms:

1) $3x$, $5x$, $7x$, $4x$, x
2) $2xy$, $7yx$, $\frac{1}{2}xy$, $-4yx$
3) $\frac{3xy^2}{7}$, $\frac{y^2x}{7}$, $\frac{2xy^2}{7}$

NOTE: Remember $xy = yx$.

The following are terms which are not alike:

$3x$, $6x^2$

$5xy$ $\frac{5x}{y}$

Like terms may be added or subtracted simply by adding or subtracting their coefficient~~al~~ parts.

$$3x + 5x = 8x$$

$$6xy - 5xy = -1xy \text{ or } -xy$$

$$5x^2y - 10x^2y = -5x^2y$$

However, $10xy^2 + 5xy$ may not be added since they have different variables.

A term is thought of as a single entity that represents one particular value. The word term is useful in allowing us to talk about parts of larger expressions.

Expressions:

There are two types of expressions:

1) Arithmetic
2) Algebraic

Arithmetic or numerical expressions are made up of constant terms added or subtracted together:

$3 + 4 - 5 \cdot (6 + 3)$ is an arithmetic expression made up of three terms; $3, 4, 5 \cdot (6 + 3)$.

An algebraic expression may contain constants and/or variable terms added or subtracted, such as:

$3y + 5 \cdot (3 + x)$ is an algebraic expression made up of two terms $3y$ and $5 \cdot (3 + x)$.

Variables

In the preceding topics we talked about algebraic expressions, such as:

$$3 + 2x$$

This algebraic expression has the value depending upon what value we assign x. For instance, if we assign x the value of 5, we see the expression has the value of 13.

$$3 + 2 \cdot (5) = 3 + 10 = 13$$

If we give x the value of 10 we see the expression has the value of 23.

$$3 + 2 \cdot (10) = 3 + 20 = 23$$

Thus the value we assign x may be changed or varied. For this reason we call x a variable. All the letters, a - z and some characters in arithmetic expressions are thought of as variables.

$2x + 5y - 2xz$ is made up of 3 variables, x, y and z.

Equations:

Equations are two expressions seperated by a basic symbol of equality (=) or inequality ($>$, $\geq$, $<$, $\leq$, $\neq$).

For Example:

$$3x - 2 = 5 - 2y$$

$$6y + 3x \geq 5 + 2z$$

Explanation of Symbols: (→ means "implies")

$>$ → Greater than

$\geq$ → greater than or equal to

$<$ → less than

$\leq$ → less than or equal to

$\neq$ → is not equal to

STRUCTURE IN ALGEBRA

Before a game is played, knowing the rules can make it more interesting and challenging. Not playing by the rules can cause frustration and confusion.

The science of Mathematics has many properties (or rules) which apply in order to work problems in an efficient and correct way.

It was only during the early nineteenth century that structure in algebra was noticed. Particularly, British mathematicians George Peacock (1791 - 1858) and Duncan Farquharson Gregory (1813 - 1817) discovered interesting facts working with algebra.

By rearranging multiplication and addition problems, the presence of a structure occurred. The effects of zero and one were revealed. Works by Peacock and Gregory, along with other students of algebra at the time, led to what is known today as:

I. Commulative Laws of Addition and Multiplication.
II. Associative Laws of Addition and Multiplication.
III. Distributive Law of Multiplication over Addition.
IV. Grouping
V. Closure
VI. Properties of Zero

VII. Properties of One

In this section, we will learn these properties and explain why they exist.

COMMUTATIVE PROPERTY OF ADDITION

If a and b are any REAL NUMBERS, the following property is true:

$$a + b = b + a$$

Remember a and b denote any two real numbers. If we choose specific numbers, say $a = 2$ and $b = 5$, we can see why the property is true. For example,

$a + b = b + a$	Property is stated.
$2 + 5 = 5 + 2$	Substitution.
$7 = 7$	Equality.

Therefore, using the operation ADDITION of two REAL NUMBERS, a + b equals the same answer as b + a. The real numbers can be added in any order.

→ OBSERVATION 1 - Closure

If two REAL NUMBERS are added together, the resulting answer is also a REAL NUMBER. Because of this, we say:

> The Commutative Property
> of Addition of Real
> Numbers is CLOSED.

No matter how many real numbers are added to each other, the result will be a real number. We could never get an answer that looks like *, #, or @. These symbols are not members of the REAL NUMBER SYSTEM.

COMMUTATIVE PROPERTY OF MULTIPLICATION

If a and b are any REAL NUMBERS, the following property is true:

$$(a)(b) = (b)(a)$$

Another way of indicating multiplication is:

$$a \cdot b = b \cdot a$$

If we choose specific numbers, say a = 3, and b = 7, we can see why the property is true. For example,

$(a)(b)$	=	$(b)(a)$	Property is stated.
$(3)(7)$	=	$(7)(3)$	Substitution.
21	=	21	Equality.

→ OBSERVATION 1 - Closure

If two REAL NUMBERS are multiplied together, the resulting answer is a REAL NUMBER. Because of this, we say:

The Commutative Property of
Multiplication is CLOSED.

→**OBSERVATION 2** - Symbolism

The reason algebra books do not ordinarily use $a \times b = b \times a$, is because the "times" symbol, ×, could be mistaken as the letter x: a variable used most often in algebra.

→ **OBSERVATION 3** - More than two Real Numbers

We have just learned that if a and b are any two REAL NUMBERS, $a + b = b + a$. Also, more than two REAL NUMBERS apply:

$$a + b + c + d = d + c + b + a$$

Consequently, if more than two REAL NUMBERS are multiplied together, equality will exist:

$$(a)(b)(c)(d) = (d)(c)(b)(a)$$

COMMUTATIVE MEANS THAT THE REAL NUMBERS CAN BE ADDED IN ANY ORDER, OBTAINING THE SAME ANSWER. LIKEWISE, REAL NUMBERS CAN BE MULTIPLIED IN ANY ORDER, OBTAINING THE SAME ANSWER.

ASSOCIATIVE PROPERTY OF ADDITION

The Commutative Property of Addition demonstrated that real numbers could be added in any order, producing the same answer.

The ASSOCIATIVE PROPERTY OF ADDITION shows that grouping real number will not change the answer.

We know that,

$$2 + 3 + 4 = 9$$

Using ( ) as grouping symbols,

$$(2 + 3) + 4 = 9$$
$$5 + 4 = 9$$
$$9 = 9$$

With different grouping,

$$2 + (3 + 4) = 9$$
$$2 + 7 = 9$$
$$9 = 9 \checkmark$$

Therefore, the Associative Property of Addition states:

Let a, b, and c be any REAL NUMBERS; then,

$$(a + b) + c = a + (b + c)$$

ASSOCIATIVE PROPERTY OF MULTIPLICATION

The Commutative Property of Multiplication demonstrated that REAL NUMBERS could be multiplied in any order, producing the same answer.

The ASSOCIATIVE PROPERTY OF MULTIPLICATION shows that grouping real numbers will not change the answer.

We know that,

$$(2)\ (3)\ (4) = 24$$

Also, it is true,

$$\{ (2)\ (3) \}(4) = 24$$
$$(6)\ (4) = 24$$
$$24 = 24 \checkmark$$

Or, grouping in another way,

$$(2)\ \{ (3)\ (4) \} = 24$$
$$(2)\ (12) = 24$$
$$24 = 24 \checkmark$$

Therefore, the ASSOCIATIVE PROPERTY OF MULTIPLICATION states:

Let a, b, c be any REAL NUMBERS, then:

$$\{ (a)(b) \} (c) = (a) \{ (b)(c) \}$$

Another way of expressing the property:

$$(ab)c = a(bc)$$

→ OBSERVATION 1 - Grouping

The ASSOCIATIVE properties of ADDITION and MULTIPLICATION deal with grouping.

Does CLOSURE exist? Yes. No matter how you group REAL NUMBERS demonstrating the Associative properties of addition and multiplication, respectively, a real number will result as the answer.

PROPERTIES OF ZERO

A simple math problem would be:

$$6 \div 2 = 3 \quad \text{or} \quad \begin{array}{r} 3 \\ 2\overline{)6} \\ \underline{6} \\ 0 \end{array} \quad \text{because } 3 \cdot 2 = 6$$

Utilizing zero,

$$0 \div 3 = 0 \quad \text{or} \quad \begin{array}{r} 0 \\ 3\overline{)0} \\ \underline{0} \\ 0 \end{array} \quad \text{because } 0 \cdot 3 = 0$$

What if division by zero is introduced:

$$3 \div 0 = ? \quad \text{or} \quad \begin{array}{r} ? \\ 0\overline{)3} \end{array} \quad \text{because } ? \cdot 0 = 3$$

The last example problem is impossible to work. There is no number you could multiply by zero and obtain an answer of 3.

→ OBSERVATION 1 - Division by Zero

Division by zero is not permissable.

Likewise, 0 ÷ 0 or 0/0 is not possible. These are ambiguous expressions; i.e. having more than one meaning. One answer for these expressions cannot be defined.

→ OBSERVATION 2 - Additive Identity

Any number added to, or subtracted from zero, is left unchanged. Therefore, zero is called the ADDITIVE IDENTITY.

$$a = a + 0 = a - 0 = a$$

The REAL NUMBER, a, does not loose its identity when added to, or subtracted from zero.

→ OBSERVATION 3 - Additive Inverse

Knowing the ADDITIVE IDENTITY, the ADDITIVE INVERSE can be derived. By definition, "inverse" means "that which is opposite." The opposite of addition is subtraction.

To find the additive inverse of any REAL NUMBER, the additive identity must be known.

{ ANY REAL NUMBERS } + { ADDITIVE INVERSE } = ADDITIVE IDENTITY

Substituting the REAL NUMBER, a, into the above formula:

$$\{ a \} + \{ ? \} = 0$$

What could be added to a, in order to obtain the additive identity, 0?

$$\begin{aligned} \{ a \} + \{ -a \} &= 0 \\ a - a &= 0 \\ 0 &= 0 \checkmark \end{aligned}$$

Therefore, the additive inverse of a, is -a. An example using a specific number would be:

Q: What is the additive inverse of 1/3?

A: { ANY REAL NUMBERS } + { ADDITIVE INVERSE } = ADDITIVE IDENTITY

$$\begin{aligned} \{ 1/3 \} + \{ \quad \} &= 0 \\ 1/3 + (-1/3) &= 0 \\ 0 &= 0 \end{aligned}$$

The additive inverse of 1/3, is -1/3.

PROPERTIES OF ONE

It has been shown that the properties of zero exist with reference to addition and its inverse, subtraction. Also, division by zero is not permissable.

"The unique properties of ONE exist in reference to multiplication, and its inverse, division."

If a is any REAL NUMBER, then:

$$a = a \cdot 1 = 1a = a1 = a = \frac{a}{1}$$

Any REAL NUMBER, excluding zero, divided by itself, results in ONE.

$$\frac{a}{a} = 1 \text{ provided } a \neq 0$$

→ OBSERVATION 1 - Multiplicative Identity

It can be seen that any REAL NUMBER, multiplied by ONE, does not loose its identity. Therefore, the MULTIPLICATIVE IDENTITY is ONE.

→ OBSERVATION 2 - Multiplicative Inverse

Since the MULTIPLICATIVE IDENTITY is ONE, we can derive the MULTIPLICATIVE INVERSE.

{ ANY REAL NUMBER } · { MULTIPLICATIVE INVERSE }= MULTIPLICATIVE IDENTITY

Substituting the REAL NUMBER, a, into the above formula,

$$\{ a \} \cdot \{ ? \} = 1$$

$$a \cdot \frac{1}{a} = 1$$

$$\frac{a}{a} = 1$$

To find the multiplicative inverse of a, remember the opposite of multiplication, is division. Therefore, the multiplicative inverse of a, is $\frac{1}{a}$. This is because (a) $(\frac{1}{a})$ equals the multiplicative identity, ONE.

The multiplicative inverse of $\frac{1}{a}$ would be a, because $(\frac{1}{a}) \cdot (a)$ equals ONE.

→ OBSERVATION 3 - Reciprocal

Another word, or term, should be introduced at this point. When referring to multiplicative inverse, we often say, THE RECIPROCAL OF. For example,

The (multiplicative inverse) of a is $\frac{1}{a}$.

The (reciprocal) of a is $\frac{1}{a}$.

The (multiplicative inverse) of $\frac{1}{a}$ is a.

The (reciprocal) of $\frac{1}{a}$ is a.

Solve the following problems. List each axiom or property used.

1) $x + 2 = 5$

2) $2x + 4 = 6$

3) $\frac{x}{2} = 4$

4) $3x - 1 = 8$

5) $\frac{4x}{3} + 2 = -6$

Chapter II - Algebra (Part II)

1. Exponents
2. Polynomials
3. Multiplying Two Expressions
4. Factoring
5. Algebraic Fractions

EXPONENTS

The concept of exponents is very simple. One definition could be expressed as:

If $a^1 = a$, with all $n > 1$, then:

$$a^n = \underbrace{a \cdot a \cdot \ldots \cdot a}_{n \text{ factors}}$$

Examine $a^1 = a$. This means that a constant, or variable base having an exponent of 1 (a power of one), remains unchanged. However, as exponents increase, greater than one, the base constant will assume a new value. For example:

$$3^{1.1} = 3.348$$

$$3^{1.5} = 5.196$$

$$3^2 = 9$$

Base variables also take new values. For example:

$$x^2 = x \cdot x$$

$$y^3 = y \cdot y \cdot y$$

$$z^{.5} = z^{\frac{1}{2}} = \sqrt{z}$$

Laws of Exponents

Algebraic equations may contain variables and constants. Multiplication of constants is simple. However, certain laws relating to exponents are necessary before multiplication or division is performed.

Rule I

If x is any real number, then:

$$x^1 = x$$

$$x^0 = 1$$

Rule II

If $x \neq 0$, with m and n real numbers, then:

$$x^m \cdot x^n = x^{m+n}$$

Rule III

If x, m, and n are any real numbers:

$$(a^m)^n = a^{m \cdot n}$$

"Power Theorem"

Rule IV

If x, m, and n are any real numbers, with $x \neq 0$, then:

$$\frac{x^m}{x^n} = x^{m-n} = \frac{1}{x^{n-m}}$$

Rule V

$0^0 =$ → undefined

When dealing with radical signs; square roots, cube roots, etc., the signs need to be changed into an exponent denoting the particular root. For example:

$$\sqrt{x} = (x)^{\frac{1}{2}}$$

$$\sqrt[3]{z} = (z)^{\frac{1}{3}}$$

$$\sqrt[4]{y^2} = (y)^{\frac{2}{4}}$$

These examples are obtained from the following rule:

Rule VI

$$\boxed{\sqrt[n]{x^m} = x^{\frac{m}{n}}}$$

Simplify - Give Results in Positive Powers:

1) $\frac{1}{x^{-3}}$

2) $2x^{-2}$

3) $x^{2} \cdot x^{4}$

4) $\frac{x^{3} \cdot x^{4}}{x^{2}}$

5) $(2)^{-3}$

6) $\frac{x^{-2}\ x^{-4}}{x^{-3}}$

7) $x^{-2}\ y^{4} \cdot x^{+3}\ y^{-6}$

8) $\frac{3x^{2}\ y^{3}\ z^{2}}{6x^{4}\ y^{-2}}$

9) $\frac{660\ x^{-2}\ y^{-3}}{1320\ x^{-4}\ y^{-2}}$

10) $\frac{3(xy)^{2} \cdot x}{x^{2}\ y^{2}}$

Perform the following operations utilizing laws of exponents:

1) $x^2 \cdot x^3$

2) $y^4 \cdot y^{-2}$

3) $xy^2z\ x^3y^{-5}$

4) $\frac{3x^2y^4z^{-3}}{6xy^5}$

5) $\frac{z^{\frac{1}{2}}}{z^4}$

6) $\frac{2x^1}{x^1}$

7) $(4x^3)^2$

8) $\frac{a^2b^3c^4}{a\ b\ c}$

9) $\frac{10x}{x}$

10) $\frac{27xyz}{9x}$

Simplify:

1) $\sqrt[4]{x^2}$

2) $\sqrt{9x^2}$

3) $x\sqrt{x^3}$

4) $\sqrt[3]{27x^6}$

5) $\dfrac{x^{\frac{3}{2}}}{x^{\frac{1}{8}}}$

6) $\sqrt[3]{8x^5 y^4}$

7) $2\sqrt[3]{y^4}$

8) $x^{\frac{2}{3}}\sqrt[3]{x}$

9) $\dfrac{1}{\sqrt{x^4}}$

10) $\dfrac{\sqrt{x^5}}{\sqrt[3]{x^2}}$

POLYNOMIALS

Definition

The word polynomial is used many times. What exactly is a polynomial? By definition, a polynomial is "many terms". More specifically, it is an algebraic function with a sum of two or more terms, consisting of one or more variables. The correct mathematical explanation is:

A polynomial function denoted $P_n(x)$, is:

$$P_n(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0 ,$$

where n → a natural number or zero with $a_n \neq 0$. Also, if n is non-negative, n is called its degree

a_i → coefficients beginning with an initial (i) coefficient

a_n → the leading coefficient

To further explain, a polynomial is an algebraic expression with only positive integral (of the set of integers) powers of a variable. If the algebraic expression is a fraction, it is a polynomial only if there are no variables in the denominator of the fraction.

EXAMPLES OF POLYNOMIALS

$ax^2 + bx + c$	$\frac{2x + 3}{4}$
$2x^2 - 3x + 5$	$-6y^2 - 2y + 3$
$6x$	$\frac{7x^2 - 5}{10}$
$\frac{1}{4}y^2 + 2$	

The algebraic expression of a type, $\frac{3x^2 - 2}{4x}$ is not a polynomial because a variable appears in the denominator.

Once we have established that a given algebraic expression is a polynomial, each expression has an identifying name. Such as:

Polynomial	Identifying Name
$6x$	Monomial → One Term
$3y^2 - 2$	Binomial → Two Terms
$x + 3z - 4y$	Trinomial → Three Terms
7	Monomial → One Term, also called "a constant term".

The degree of a polynomial is the same as the highest power variable. For example:

Polynomial	Identifying Name	Degree
$6x$	Monomial	First
$2x^2 - 5x + 4$	Trinomial	Second
$3x^3 + 5$	Binomial	Third (Cubic)
4	Monomial	Zero because $x^0 = 1$, therefore $4x^0 = 4$

If polynomials have powers higher than degree three, the algebraic expression is known as:

Polynomial	Degree
$x^4 + 5x^3 + 2x$	Fourth (Quartic)
$7x^5$	Fifth (Quintic)
$x^6 - 10x^4 + 3x^2$	Sixth (Sixtic)
$\vdots$ ↓ and so on	

Identify the following polynomials by name and degree:

1) $2x^2 - 6$

2) $5z$

3) $2x^3 - x^2 - 5$

4) $x^6 - 10x^4 + 3x^2$

Create an algebraic expression for each term requested:

1) Monomial of Second Degree

2) Trinomial of First Degree

3) Binomial of Sixth Degree

4) Trinomial of Third Degree

MULTIPLYING TWO EXPRESSIONS

Multiplying two or more expressions together is important before we go further in techniques of factoring. If one of the two expressions contains only one term, multiplication is easily handled by use of the distributive law of multiplication.

$$A \cdot (B + C) = A \cdot B + A \cdot C$$

For instance, multiply the following expressions:

$$3x \cdot (8x^2 + 10x + 7)$$

$$3x \cdot (8x^2) + 3x\,(10x) + 3x \cdot (7)$$

$$24x^3 + 30x^2 + 21x$$

A problem arises when both expressions contain two or more terms. Such as:

$$2x + 5 \qquad 3x - 2$$

Multiplying these type of expressions is dealt with by taking the first term of one expression and multiplying it times all the terms of the second. Next, take the second term of the first expression and multiply it times all terms of the second. Then, take the third and so on until we use each term of the first expression. Finally, collect like terms.

Multiply the following expression:

$$(2x + 5) \cdot (3x - 2)$$

$$2x \cdot (3x - 2) + 5 \cdot (3x - 2)$$

$$6x^2 - 4x + 15x - 10$$

$$6x^2 + 11x - 10$$

Another commonly used method is to write one expression directly below the other. Then take the first term of the bottom and multiply it times all terms in the upper expression. Next, take the second term of the bottom and multiply it times all the terms of the upper and place it in its proper position for easy addition. Add the like terms.

Multiply the following expression:

$$\begin{array}{rrr}
 & 2x & + 5 \\
 & 3x & - 2 \\
\hline
6x^2 & + 15x & \\
 & - 4x & - 10 \\
\hline
6x^2 & + 11x & - 10
\end{array}$$

Multiply the following expressions.

1) $(x + 1) \cdot (2x + 1)$

2) $(2x + 4) \cdot (3x + 7)$

3) $(x - 1) \cdot (x + 4)$

4) $(x + 2) \cdot (x - 2)$

5) $(x + 4)^2$

6) $(2x - 4) \cdot (x - 2)$

7) $(x^2 + 2x + 1) \cdot (x + 2)$

8) $4 \cdot (x + 1) \cdot (2x + 2)$

9) $(x - 4) \cdot (3x - 2)$

10) $(x + 2) \cdot (x - 2) \cdot (x + 3)$

FACTORING

An Introduction

The fundamental theorem of arithmetic says that the prime numbers are building bricks from which all other integers are multiplicatively made.[1]

From this statement, we can begin explaining the important concept of factoring, in arithmetic through algebra.

Prime Numbers

Prime numbers are defined as any number, $n > 1$, evenly divisable only by itself and the number 1.

Examples of prime numbers are:

$\{2, 3, 5, 7, 11, 13\}$

there are more past the number 13. These numbers are prime because they comply with the required definition. For instance,

7 is prime because only the number 1 and 7 itself will evenly divide into 7.

Can you locate more prime numbers?

---

[1] Howard Eves, An Introduction to the History of Mathematics, Third Edition, Holt, Rinehart, Winston, New York, 1969, p. 149.

Product of Primes

We begin our explanation by an example:

$$\begin{array}{rcc} 12 = & 6 & \cdot\ 2 \\ & \wedge & \downarrow \\ = & 3 \cdot 2 & \cdot 2 \end{array}$$

The number 12 has been written first as $6 \cdot 2$. Then, the product of $6 \cdot 2$ could be broken down into $3 \cdot 2 \cdot 2$. This format is called the "product of primes".

Even though the number 12 is not a prime, it can be broken down into a "product of primes" -- multiplied together, the result is again, 12.

This example follows the fundamental theorem of arithmetic. And from this concept, we introduce the "breaking down" of more complex numbers, including variables. This "breaking down" process is called Factoring.

Factoring Monomials

By definition, a monomial is a polynomial consisting of one term, whether a constant (having no variables), or a term consisting of a number "times" a variable.

Examples of monomials are:

5

$2x$

$-7y^2$

$\frac{1}{4}xyz$

$-3x^2y$

$5z$

Suppose we were asked to "factor" an x out of each example listed above. Is it possible?

MONOMIAL	FACTOR X OUT
5	No x exists
$2x$	$x\ (2)$ or $2\ (x)$
$-7y^2$	No x exists
$\frac{1}{4}xyz$	$x\ (\frac{1}{4}yz)$ or $(\frac{1}{4}yz)\ x$
$-3x^2y$	$x\ (-3xy)$
$5z$	No x exists

If an x appears in the monomial, we can "factor" out an x. Multiplying the x "times" the quantity in parenthesis, will result in the original monomial.

Factoring Binomials

By definition, a binomial is a polynomial consisting of two terms.

Examples of Binomials:

$5 + 2x$

$10 + 5y$

$4x + 2yx$

$3y - 9zy$

Binomials can be factored based on the distributive law (from "Structure in Algebra"). Looking further at the above examples, can any be factored? In other words, can a constant, or variable, or both, be factored out of each binomial? Study the following:

BINOMIAL	FACTORED
$5 + 2x$	Cannot be factored
$10 + 5y$	$5 (2 + y)$
$4x + 2yX$	$2x (2 + y)$
$3y - 9zy$	$3y (1 - 3z)$

Our table shows the binomials which could be factored, or "broken down" into lowest terms. The term "factored out" of the original expression, is called the common factor. Therefore, to check, apply the distributive property: Multiply the common factor times each term within the parenthesis.

Factoring Trinomials

A trinomial, by definition, is a polynomial composed of three terms.

Examples of Trinomials:

$5 + x + z$

$4x + 2y + 6z$

$6xyz^2 - 3z + 2$

Can the examples listed above be factored (reduced to lowest terms)?

TRINOMIAL	FACTORED
$5 + x + z$	Cannot be factored
$4x + 2y + 6z$	$2\ (2x + y + 3z)$
$6xyz^2 - 3z^3 + 2z^2$	$z^2\ (6xy - 3z + 2)$

By applying the distributive property and properties of exponents, the factored versions can be checked.

Factor Out the Common Term:

1) $x^2 + 5x$

2) $a^2b^2 + 3ab$

3) $15x^2y^2 - 30xy + 25x^2y$

4) $6a^2b - 16ab$

5) $101A^2 - 501A$

6) $20A^2B^3C^2 + 15\ ABC + 25AB^2C^2$

7) $5x + 5$

8) $21x^5y^4z^3 + 7x^4y^3z^2 - 49x^3y^2z^2$

9) $45x^2y^3 + 81x^2y^2 - 99x^2yz$

10) $15x^{\frac{7}{2}} - 3x^{\frac{5}{2}} - 21x^3$

↓

Hint: Express x^3 in terms of halves, i.e., $x^4 = x^{\frac{8}{2}}$

Factoring Quadratic Trinomials

We have just learned that a trinomial is a polynomial consisting of three terms. A quadratic trinomial is a three-termed polynomial with the highest power variable of degree 2. Therefore, the word quadratic implies second-degree.

Usually, quadratic trinomials (often called "quadratics" only), are factored into the product of two binomials. For example:

Quadratic		Factored Form
$x^2 + 8x + 15$	$=$	$(x + 5)(x + 3)$

In the previous section, trinomials were factored by "taking out" a common term, variable, or constant. Then, the original trinomial was obtained by applying the distributive law. Thus,

$$2x + 2yz + 6y = 2(x + yz + 3y) = 2x + 2yz + 6y$$

With a quadratic trinomial, the process of factoring involves the following:

Step I: $x^2 + 8x + 15$ Expression must be in standard polynomial form, i.e., descending powers.

Step II:

1st Term

$$x^2 + 8x + 15 = (x + 5)(x + 3)$$

3rd Term

The product of $x \cdot x$ must equal the first term, x^2. The product of $(+5)(+3)$ must equal the third term, $+15$. The middle term is obtained by:

Step III: (x + 5) (x + 3)

↑___↑

↑_____+_____↑

↓

Middle Term

Adding the product of (+5 (x) to the product of (x) (+3).

The "signs" of numbers multiplied are used. For instance, (+5) (x) denotes a "positive 5" times a positive x. Looking further at the same problem:

$$x^2 + 8x + 15$$

↓

(+3) (+5) = +15

(-3) (-5) = +15

(+15) (+1) = +15

(-15) (-1) = +15

Notice there are several "factors" of 15, or +15 (positive 15). How do we know that (+3) (+5) = +15 is the correct set of factors?

When choosing the "factors" or "multiples" of the third term, the same two factors, added, must equal the middle term.

Example:

$$x^2 + 8x + 15 \neq (x + 15)\,(x + 1)$$

Even though (+15) (+1) equals +15, the middle term is:

$$(\underbrace{x + \underbrace{15)\,(x}_{+} + 1}_{=})$$

$$(+15)\,(x)$$

$$+$$

$$(x)\,(+1)$$

$$=$$

$$15x + 1x = \boxed{16x}$$

The middle term
should equal 8x

Consider another example:

$$x^2 + 6x + 8$$

In order to factor:

Step I: Is $x^2 + 6x + 8$ in standard polynomial form? Yes.

Step II: What are the factors of x^2? $x \cdot x$.

Therefore, the factors of the first term are $x \cdot x$.

What are the factors of the third term, + 8? Several choices:

$$(+4)(+2) = +8$$

$$(-4)(-2) = +8$$

$$(+8)(+1) = +8$$

$$(-8)(-1) = +8$$

However, the only set of factors that will add to the "middle term" of +6x, are:

$$(+4)(+2)$$

Step III: $x^2 + 6x + 8 = (x + 4)(x + 2)$

also written as:

$$x^2 + 6x + 8 = (x + 2)(x + 4)$$

More Advanced Quadratic Trinomial

A quadratic trinomial is also of the type:

$$8x^2y^2 + 16xyz + 6z^2$$

In previous examples, we were only concerned with the "factors" or "multiples of the third term. When a quadratic began as x^2....., the multiples of a first term like x^2 were easy: $x \cdot x$. More advanced quadratics require investigating both the first term and third term.

When looking at $8x^2y^2 + 16xyz + 6z^2$, notice that a common factor of 2 could be "taken out" of each term. For instance,

$$8x^2y^2 + 16xyz + 6z^2 = 2(4x^2y^2 + 8xyz + 3z^2)$$

However, factoring should always produce an expression in lowest terms. And, when the expression has three terms and a variable, or variables, of highest power 2, this indicates a quadratic trinomial.

Study the example and the multiples of the first and third term:

$$8x^2y^2 + 16xyz + 6z^2$$

$$\downarrow \qquad\qquad \downarrow$$

$$(+2xy)\ (+4xy) \qquad (+2z)\ (+3z)$$

$$(-2xy)\ (-4xy) \qquad (-2z)\ (-3z)$$

$$(+8xy)\ (+1xy) \qquad (+6z)\ (+1z)$$

$$(-8xy)\ (-1xy) \qquad (-6z)\ (-1z)$$

The correct factored version is:

$$8x^2y^2 + 16xyz + 6z^2 = (2xy + 3z)(4xy + 2z)$$

Check the result by multiplying factors.

Factor the Following

1) $x^2 + 3x + 2$

2) $x^2 - 3x + 2$

3) $x^2 - 9x + 18$

4) $x^2 - x - 6$

5) $6 - 7x - x^2$

6) $3x^2 + x - 2$

7) $7x^2 - 55x - 8$

8) $9x^2 + x + 4$

9) $6x^2 + 1x + 4$

10) $20x^2 - 19x - 6$

Multiplying Binomials - Simple Method

In the previous section, we learned that a quadratic trinomial could be factored into the product of two binomials:

$$x^2 - 4x - 12 = (x + 2)(x - 6)$$

The factored version is true because:

```
   1st term
 |_________|
 |         |
(x + 2) (x - 6)
     |_________|
      3rd term
```

$$\text{1st term} = (x)(x) = x^2$$

$$\text{3rd term} = (+2)(-6) = -12$$

To obtain the middle term:

```
(x + 2) (x - 6)
 |    |___|    |
 |      +      |
 |_____________|
        ↓
```

$$(+2)(x) + (x)(-6) = 2x + (-6x) = -4x$$

An easier way to multiply binomials, and even polynomials of more terms, is a vertical type multiplication:

$$(x + 2)(x - 6) \rightarrow \begin{array}{rrr} x & +\ 2 & \\ \uparrow & \times\ \ \uparrow & \\ \hline x & -\ 6 & \\ \hline x^2 & +\ 2x & \\ & -\ 6x & -12 \\ \hline x^2 & -\ 4x & -12 \end{array}$$

Another example:

$$(x - 3)\ (x + 4) \rightarrow \begin{array}{l} x - 3 \\ \uparrow \nwarrow\!\!\!\!\nearrow \uparrow \\ \underline{x + 4} \\ x^2 - 3x \\ \underline{\quad + 4x \quad - 12} \\ x^2 + 1x - 12 \end{array}$$

Factoring - Difference of Two Squares

The difference between two squares is a special case in factoring. Though the process of factoring is the same, a set rule exists:

$$\boxed{(a)^2 - (b)^2 = (a + b)(a - b)}$$

Perfect squares which occur frequently in algebra can be easily recognized.

$4 = (2)^2$ $36 = (6)^2$

$9 = (3)^2$ $49 = (7)^2$

$16 = (4)^2$ $64 = (8)^2$

$25 = (5)^2$ ....and so on....

Also, variables can be perfect squares:

$$x^2 = (x)^2$$

$$y^2 = (y)^2$$

$$x^2z^2 = (xz)^2$$

Sometimes, perfect squares exist which are a combination of constants and variables:

$$4x^2 = (2x)^2$$

$$25x^2y^2 = (5xy)^2$$

$$16z^2 = (4z)^2$$

$$\tfrac{1}{4}x^2 = (\tfrac{1}{2}x)^2$$

If a quadratic binomial expression contains a perfect square, it should be further investigated to see if factoring by rule is necessary:

EXAMPLE: Factor $x^2 - 4$

$$x^2 - 4 = (x)^2 - (2)^2$$

$$= (x + 2)(x - 2) \quad \text{By Rule}$$

$$\text{or} = (x - 2)(x + 2) \quad \text{Commutative Property}$$

EXAMPLE: Factor $9 - 25z^2$

$$9 - 25z^2 = (3)^2 - (5z)^2$$

$$= (3 + 5z)(3 - 5z) \quad \text{By Rule}$$

$$\text{or} = (3 - 5z)(3 + 5z) \quad \text{Commutative Property}$$

EXAMPLE: Factor $36x^2 - 9z^2$

$$36x^2 - 9z^2 = (6x)^2 - (3z)^2$$

$$= (6x - 3z)(6x + 3z)$$

EXAMPLE: Factor $3x^2 - 16$

$3x^2 - 16$ = Cannot Be Factored

Each term has to be a perfect square before the factoring rule can be applied.

An interesting observation is the sum of two squares, $a^2 + b^2$, cannot be factored.

Factoring - Difference of Two Cubes

A binomial expression consisting of the difference between two cubes is factored according to the following rule.

$$\boxed{(a)^3 - (b)^3 = (a - b)\,(a^2 + ab + b^2)}$$

Study the examples:

EXAMPLE: Factor $x^3 - y^3$

$$\begin{array}{ccccccccccc} x^3 - y^3 = & (x)^3 & - & (y)^3 & = & (x & - & y) & (x^2 & + xy & + y^2) \\ & \downarrow & & \downarrow & & \downarrow & & \downarrow & \downarrow & \downarrow & \downarrow \\ & (a)^3 & - & (b)^3 & = & (a & - & b) & (a^2 & + ab & + b^2) \end{array}$$

EXAMPLE: Factor $27 - 8z^3$

$$\begin{array}{cccc} 27 - 8z^3 = & (3)^3 & - & (2z)^3 \\ & \downarrow & & \downarrow \\ & a & & b \end{array}$$

The rule states: $(a)^3 - (b)^3 = (a - b)\,(a^2 + ab + b^2)$

Therefore: $(3)^3 - (2z)^3 = (3 - 2z)\,(9 + 6z + 4z^2)$

Factoring - Sum of Two Cubes

A binomial expression consisting of the sum between two cubes is factored according to the following rule:

$$\boxed{(a)^3 + (b)^3 = (a + b)(a^2 - ab + b^2)}$$

Study the following examples:

EXAMPLE: Factor $x^3 + y^3$

$$x^3 + y^3 = (x)^3 + (y)^3 = (x + y)(x^2 - xy + y^2)$$

$$\downarrow \qquad \downarrow \qquad \downarrow \quad \downarrow \quad \downarrow \qquad \downarrow \qquad \downarrow$$

$$(a)^3 + (b)^3 = (a + b)(a^2 - ab + b^2)$$

EXAMPLE: Factor $27 + 8z^3$

$$27 + 8z^3 = (3)^3 + (2z)^3$$

$$\downarrow \qquad \downarrow$$

$$a \qquad b$$

The rule states: $(a)^3 + (b)^3 = (a + b)(a^2 - ab + b^2)$

Therefore: $(3)^3 + (2z)^3 = (3 + 2z)(9 - 6z + 4z^2)$

ALGEBRAIC FRACTIONS

An Introduction

In the beginning, we learned the definition of a fraction and how to reduce fractions to lowest terms. Algebraic fractions are composed of a numerator and denominator, but instead of number parts only, algebraic fractions contain variables and constants. For example:

$$\frac{3x^2}{4y} \qquad \frac{(x - 1)(x + 2)}{x + 5} \qquad \frac{3xy^2 + 4}{6z}$$

Before algebraic fractions can be added, subtracted, multiplied, or divided, the numerator and denominator must be factored into lowest terms.

Addition

Simple algebraic fractions can be added without many steps.

Example 1:

$$\frac{2x}{4y} + \frac{6x}{4y} = \frac{2x + 6x}{4y} = \frac{8x}{4y}$$

In the example, the common denominator is 4y. The same rules apply; when there is a common denominator, add the numerators if they consist of like terms. Consider an example more difficult:

Example 2:

$$\frac{2x}{3z} + \frac{2y}{5z}$$

In this example, a common denominator does not exist in present form -- we have to find a common denominator. What number and variable will both 3z and 5z divide evenly?

Answer: 15z. Therefore, we "increase" each term to an equivalent fraction of higher terms, with a common denominator of 15z:

$$\frac{(5)}{(5)} \cdot \frac{2x}{3z} + \frac{2y}{5z} \cdot \frac{(3)}{(3)}$$

$$= \frac{10x}{15z} + \frac{6y}{15z}$$

$$= \frac{10x + 6y}{15z}$$

Once a common denominator and equivalent fractions are obtained, the numerators can be added.

However, in algebraic fractions, if the numerators consist of unlike terms, we can only indicate addition by "+" signs.

Subtraction

As in addition, a common denominator must exist before subtracting fractions. And numerators should consist of like terms.

Example 1:

$$\frac{3xy}{4z} - \frac{2xy}{4z} = \frac{1xy}{4z}$$

Example 2:

$$\frac{5yz}{2x} - \frac{10yz}{3y}$$

Lowest common denominator would be: 6xy.

$$\frac{(3y)}{(3y)} \cdot \frac{5yz}{2x} - \frac{10yz}{3y} \cdot \frac{(2x)}{(2x)}$$

$$\frac{15y^2z}{6xy} - \frac{20yzx}{6xy}$$

$$\frac{15y^2z - 20yzx}{6xy}$$

Because the numerators consist of unlike terms, we cannot subtract, 15 - 20.

Multiplication

Multiplying algebraic fractions involves the same process as other simple fractions. However, with more difficult fractions, numerators and denominators should be factored.

An algebraic fraction which doesn't really require factoring, before multiplying, would be of the type:

$$\frac{2x}{6z} \cdot \frac{3y}{10xy} = \frac{6xy}{60xyz}$$

Then, the final answer can be reduced to lowest terms:

$$\frac{6\cancel{xy}}{60\cancel{xy}z} = \frac{1}{10z}$$

An example that does require factoring before multiplication, is:

$$\frac{x^2 - 4x - 12}{2z} \cdot \frac{7y}{x^2 + 6x + 8}$$

$$= \frac{\cancel{(x + 2)}(x - 6)}{2z} \cdot \frac{7y}{\cancel{(x + 2)}(x + 4)}$$

$$= \frac{7y\,(x - 6)}{2z\,(x + 4)} \quad \text{Final Form}$$

Division

Again, factoring of terms before dividing can make the process very simple. Consider:

$$\frac{7xyz}{x^2 + 4x + 3} \div \frac{2y}{x^2 - x - 2}$$

$$= \frac{7xyz}{x^2 + 4x + 3} \cdot \frac{x^2 - x - 2}{2y}$$

$$= \frac{7xyz}{\cancel{(x + 1)}(x + 3)} \cdot \frac{(x - 2)\cancel{(x + 1)}}{2y}$$

$$= \frac{7xyz\,(x - 2)}{2y\,(x + 3)} \quad \text{Final Form}$$

Evaluate:

1) $\frac{6x^2}{7y^2} \cdot \frac{3z}{4x}$

2) $\frac{x}{5} + \frac{2x}{5}$

3) $\frac{6y}{2z} - \frac{3y}{2z}$

4) $\frac{5xyz}{9x} + \frac{4}{3y}$

5) $\frac{25x}{3} \div \frac{5xy}{2}$

6) $\frac{x}{3} + \frac{y}{5} - \frac{z}{15}$

7) $\frac{5z}{x^2 - x - 6} \div \frac{2x}{x^2 + x - 2}$

8) $\frac{18xy}{z} \div \frac{2ab}{xz^2}$

9) $\frac{2}{x} - \frac{3y}{z} + \frac{5}{6}$

10) $\frac{a^2b^2c^2}{xy} \cdot \frac{x^2y^2}{5}$

Multiply the Following:

1) $(x + 2)(3x - 1)$

2) $(x + 4)(x - 4)$

3) $(2x - 5)(-x + 1)$

4) $(x + 3)(x + 2)$

5) $(2x - 1)(6x + 4)$

6) $(x - 5)(x + 5)$

7) $(x - y)(x^2 + xy + y^2)$

8) $(x + 1)(x^2 + 2x + 1)$

9) $(x - y)(x + y)$

10) $x(x + 1)(x + 3)$

Addition and Subtraction of Algebraic Fractions:

1) $\frac{1}{x} + \frac{1}{24}$

2) $\frac{1}{x} + \frac{1}{x + 1}$

3) $\frac{1}{(x + 1)} - \frac{1}{(x + 1)^2}$

4) $\frac{1}{x(x + 1)} + \frac{1}{x} - \frac{1}{x + 1}$

5) $\frac{x}{x + 1} - \frac{x + 1}{x^2 + 2x + 1}$

6) $\frac{1}{(2x + 1)} - \frac{x + 1}{2x^2 - x - 1} + \frac{x}{x - 1}$

7) $\frac{2x + 1}{x} - \frac{2}{3}x$

8) $\frac{1}{x + 1} - \frac{2}{x - 1}$

9) $\frac{2}{1 - x} + \frac{1}{x - 1}$

10) $\frac{x + 1}{x - 3} - \frac{x - 3}{x + 1}$

Multiply or Divide and Simplify:

1) $\frac{3x + 3}{x + 2} \cdot \frac{(x + 2)}{x + 1}$

2) $\frac{x + 1}{2x + 4} \cdot \frac{x + 2}{x + 1}$

3) $\frac{3x^2}{x + 1} \div \frac{x^2}{x}$

4) $\frac{x^2 + 2x + 1}{x^2 + 3x + 2} \div \frac{x^2 + 4x + 3}{x^2 - 2x - 3}$

5) $x^2 - x - 6 \div x^2 - 6x + 9$

6) $\frac{x^2 - 3x - 4}{x^2 - 4} \div \frac{x^2 - 2x - 8}{x^2 + x - 6}$

7) $\frac{2x^2 - 5x - 12}{4x^2 + 16x + 15} \div \frac{(x - 4)}{2x^2 - 3x - 20}$

8) $\frac{2x^2 - 1x - 12}{x^2 + 6x + 5} \cdot \frac{3x^2 + 11x - 4}{6x^2 + x - 1} \div \frac{x^2 - 16}{x^2 - 1}$

9) $\frac{x^3 - y^3}{x^3 + y^3} \cdot \frac{(x^2 - xy + y^2)}{x - y}$

10) $\frac{3x + 6}{3} \cdot \frac{9}{x + 2}$

Chapter II - Algebra (Part III)

1. Negative and Positive Values (Signed Numbers)
2. The Rectangular Coordinate System
3. Equations - First Degree (One Variable)
4. Linear Equations - Solving For Unknowns in Terms of Other Variables
5. Systems of Linear Equations
6. Equations - Second Degree (One Variable)
7. Inequalities - An Introduction
8. Inequalities - Constant and Variable Expressions
9. Linear Inequalities
10. Systems of Linear Inequalities
11. Quadratic Inequalities
12. Functions

NEGATIVE AND POSITIVE VALUES

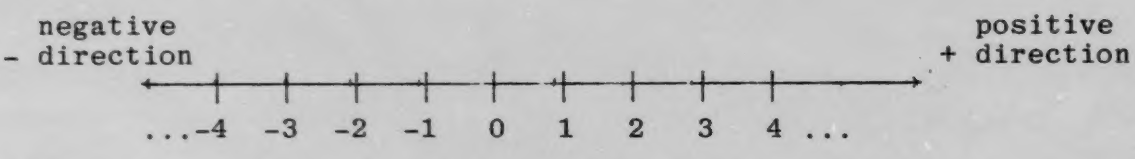
(Signed Numbers)

An Introduction

In regular arithmetic, we know that we cannot take 6 apples from 4 apples. It is not possible. However, Algebra problems may contain calculations such as:

$$4 - 6 = -2$$

By the rules of adding and subtracting positive and negative values, we can obtain an answer for 4 - 6. The numbers are taken from a number line consisting of positive and negative numbers, including zero:



In the real world, if we spoke of (4 apples - 6 apples), an answer does not exist. But speaking in terms of a number line, a negative value can result.

It is important to note that there are exceptions to this observation, which occur in the real world. Temperature, for example, is expressed in readings above zero (positive), and below zero (negative). So, in nature, a temperature of -10° is possible.

During the space shuttle launchings from Cape Canaveral, you hear a voice at mission control saying, "T minus 4 seconds and counting...." Therefore, time is being measured in terms of negative values approaching zero. After the launching, time is measured from zero, upward, in the positive direction.

The next section will provide rules of adding, subtracting, multiplying, and division of negative values.

Rules of Operation → Signed Numbers

I. Addition

Positive numbers are denoted by a + sign. Usually, numbers as we know {1, 2, 3, 4, ...} are positive numbers, but the + sign is not necessary. However, when adding or subtracting positive and negative numbers, it is important to recognize a number in terms of: 1) value, and 2) sign.

Rule 1: Adding Like Signs

A. When adding two positive numbers add the number values and "carry the sign" of the numbers.

$$(+2) + (+3) = +5 = 5$$

↓ ↓ ↓

(pos. 2) add (pos. 3) = pos. 5 = 5

II. Multiplication

When multiplying two or more signed numbers, we not only use parenthesis ( ) to denote multiplication, but parenthesis are used to "package" each signed number, keeping calculations neat. Therefore:

Rule 3: Multiplication of Like Signs

A. (Pos. No.)(Pos. No.) = Pos. Answer
↓ ↓ ↓
(+) • (+) = +

B. (Neg. No.)(Neg. No.) = Pos. Answer
↓ ↓ ↓
(–) • (–) = +

Rule 4: Multiplication of Unlike Signs

A. (Pos. No.)(Neg. No.) = Neg.Answer
↓ ↓ ↓
(+) • (–) = –

B. (Neg. No.)(Pos. No.) = Neg.Answer
↓ ↓ ↓
(–) • (+) = –

B. When adding two negative numbers, add the number values, and "carry the sign."

(- 3) + (- 7) = - 10

↓ ↓ ↓

(Neg 3) + (Neg 7) = Neg 10

Rule 2: Adding Unlike Signs

C. When adding unlike signs, first take the "difference" between the number values. The sign of answer will be the same as the sign of the larger number value.

(addend) + (addend) = Sum

↓ ↓

(- 3) + (+ 5) = + 2

or

(+ 7) + (- 10) = - 3

III. Subtraction

In order to make subtraction of signed numbers easier, we actually employ a "trick" from multiplication.

Refer to rules 3 and 4, multiplication of like and unlike signs. Then consider the example:

$$\begin{aligned} -5 \underbrace{-\ (-}_{+}8) &= -5 + 8 \\ &= +3 \end{aligned}$$

Our "trick" says that from Rule 3B, (Neg. sign) (Neg. sign) → positive answer. By multiplying the two negative signs of - (-8), we eliminated the two signs into an equivalent + sign. Then we applied Rule 2C.

Another example:

$$(+4) - (+6) = +4 - 6$$

Apply Rule 2C:

+ 4 - 6 is the same as (+ 4) + (- 6), signs are unlike, therefore, from the beginning we have:

$$\begin{aligned} (+4) \underbrace{-\ (+}\, 6) &= +4 - 6 && (-)(+) = - \\ &= (+4) + (-6) && \text{Apply Rule 2C} \\ &= -2 \end{aligned}$$

IV. Division

The rules for dividing positive and negative numbers are closely related to the rules of sign multiplication. Therefore, we list both sets of rules for the student to review and compare.

Multiplication	Division
A. (Pos.)(Pos.) = Pos.	A. $\frac{\text{(Pos.)}}{\text{(Pos.)}} = \text{Pos.}$
$(+)(+) = +$	$\frac{(+)}{(+)} = +$
B. (Neg.)(Neg.) = Pos.	B. $\frac{\text{(Neg.)}}{\text{(Neg.)}} = \text{Pos.}$
$(-)\ (-) = +$	$\frac{(-)}{(-)} = -$
C. (Neg.)(Pos.) or (Pos.)(Neg.) = Neg.	C. $\frac{\text{(Pos.)}}{\text{(Neg.)}}$ or $\frac{\text{(Neg.)}}{\text{(Pos.)}} = \text{Neg.}$
$(-)(+)$ or $(+)(-) = -$	$\frac{(+)}{(-)}$ or $\frac{(-)}{(+)} = -$

ADDITION

1) (2) + (3)

2) (- 4) + (3)

3) (7) + (- 2)

4) $(-\frac{1}{2}) + (-\frac{1}{2})$

5) $(-\frac{1}{4}) + (\frac{1}{2})$

6) $(-8) + (-\frac{1}{3})$

7) (- 2) + (4)

8) (- 5) + (3)

SUBTRACTION

1) (6) - (4)

2) (- 7) - (- 3)

3) $(\frac{1}{2}) - (-\frac{1}{2})$

4) (- 5) - (10)

5) (3) - (- 8)

6) $(\frac{1}{4}) - (5)$

7) (- 3) - (- 11)

8) (7) - (- 2)

MULTIPLICATION

1) $(3)(4)$

2) $(-\frac{1}{2})(8)$

3) $(-4)(-5)$

4) $(\frac{1}{3})(-\frac{1}{3})$

5) $(-7)(-8)$

6) $(-4)(2)$

7) $(\frac{1}{3})(\frac{1}{4})$

8) $(-1)(-3)$

DIVISION

1) $\frac{6}{3}$

2) $\frac{-4}{-3}$

3) $\frac{-12}{4}$

4) $\frac{6}{-2}$

5) $\frac{-9}{-3}$

6) $\frac{4}{-2}$

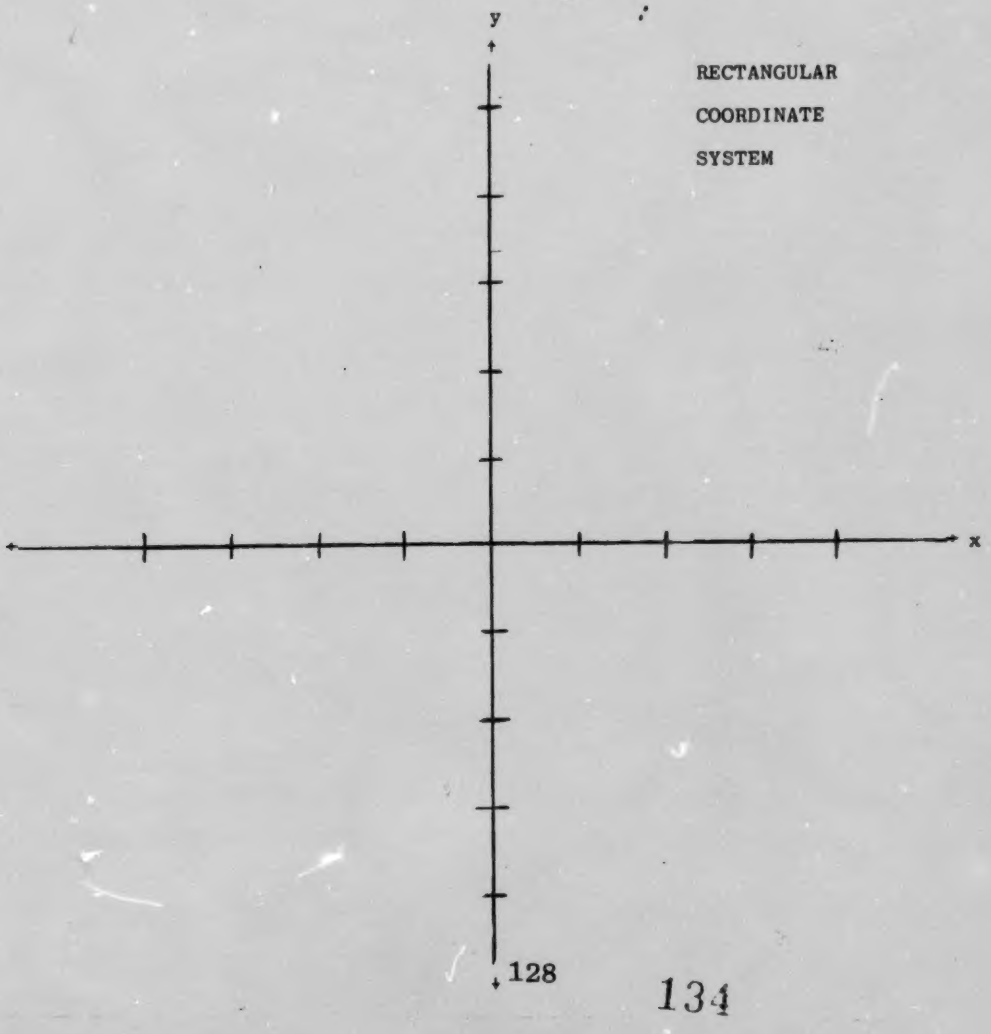
7) $\frac{2}{-3}$

8) $\frac{-10}{5}$

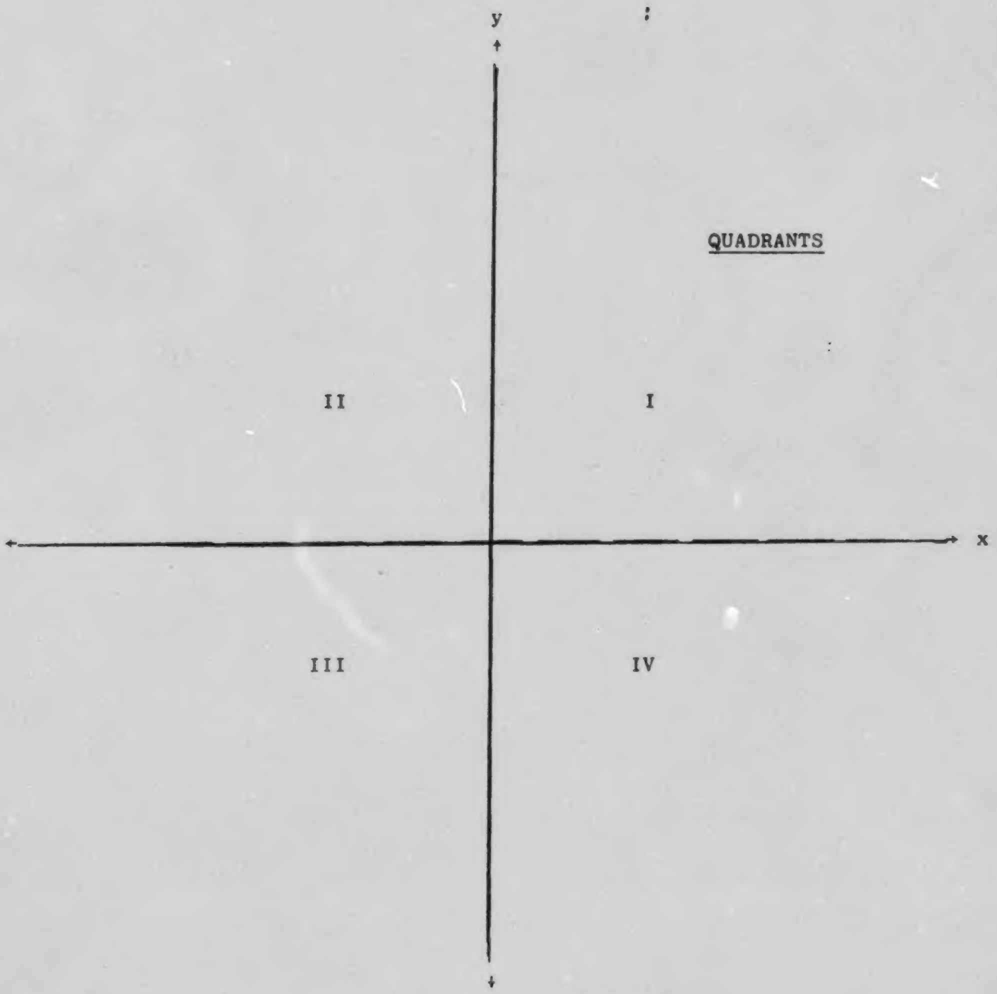
THE RECTANGULAR COORDINATE SYSTEM

Points

All of us have seen graphs at one time or another. The most commonly used graphs are made on the Rectangular Coordinate System. The system is made by constructing perpendicular number lines intersecting each other at the point zero, called the origin.


y
RECTANGULAR
COORDINATE
SYSTEM
x

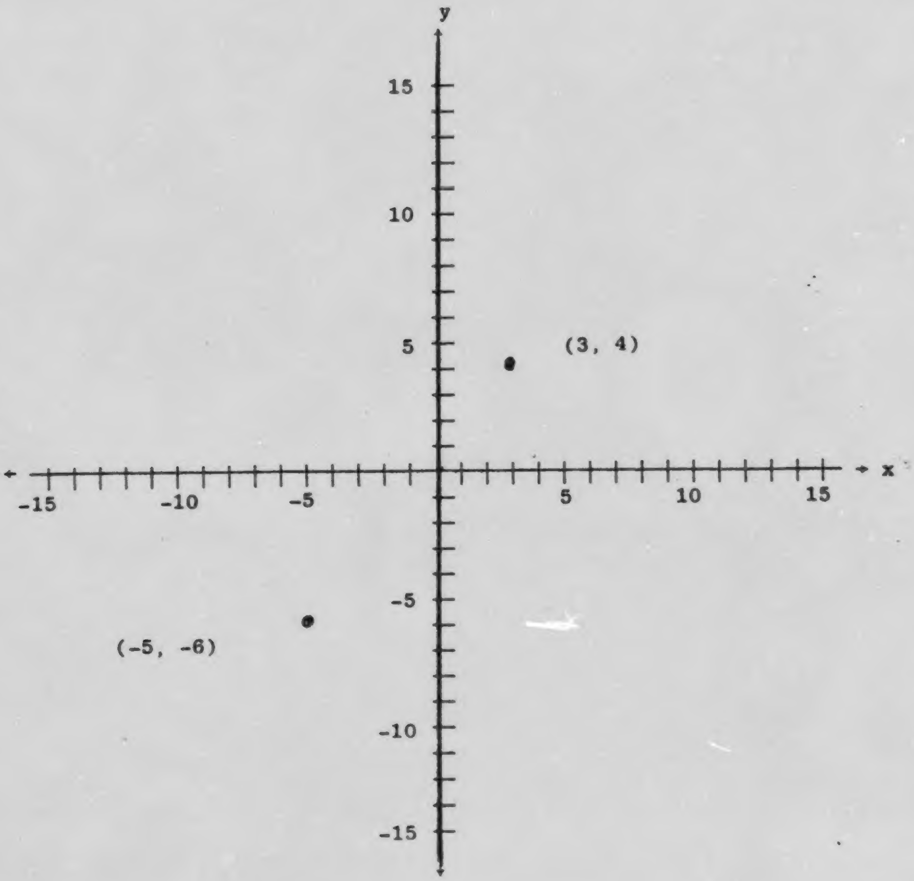
By convention, the horizontal axis is called the x axis while the vertical axis is called the y axis. The two axes divide the plane into 4 sections, each called a quadrant. The quadrants are numbered I, II, III and IV starting in the upper right quadrant and going in a counter-clockwise direction.



QUADRANTS

Any point on the Rectangular Coordinate system may be defined by a distance along the x axis, called the abscissa, and a distance along the y axis, called the ordinate.

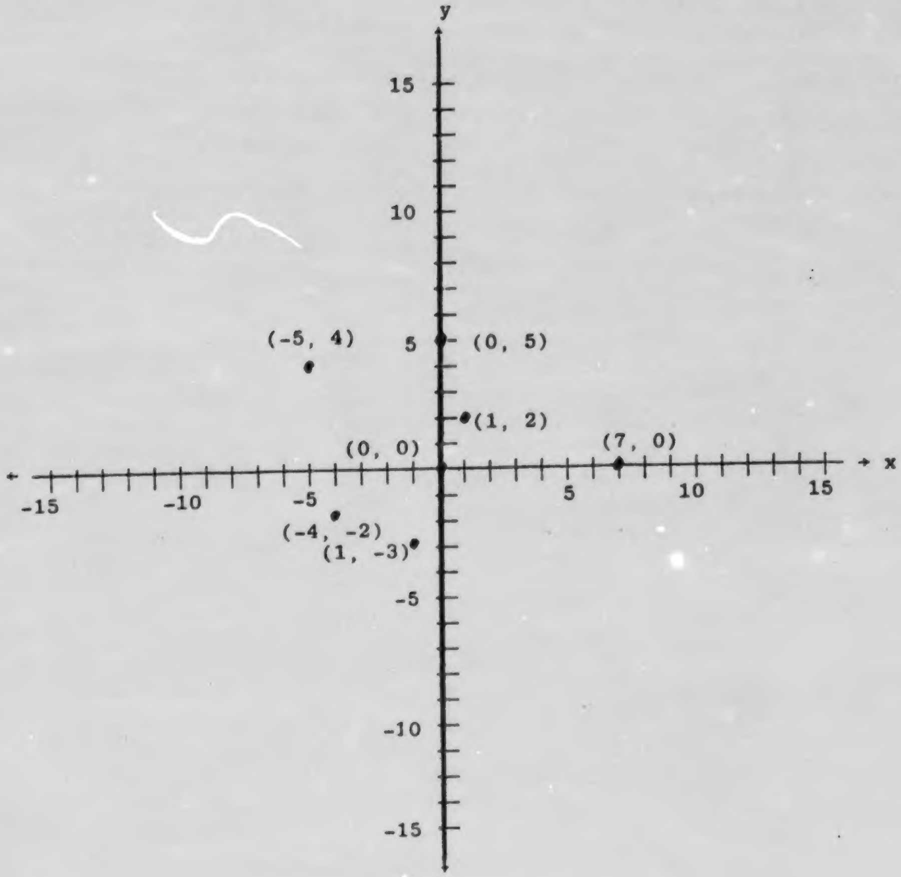
Therefore, a point which is located 3 units along the x axis and 4 units along the y axis would be the point (3,4). The point (-5,-6) would be a distance 5 units to the left of the origin on the x axis, and a distance 6 units down from the origin on the y axis:



A point on the rectangular coordinate system is composed of two numbers x and y; notation states that the point be represented as (x,y).

EXAMPLE: Plot the following points:

A = (1,2)
B = (1,-3)
C = (-4,-2)
D = (-5,4)
E = (0,5)
F = (7,0)
G = (0,0)



Rectangular Coordinate System

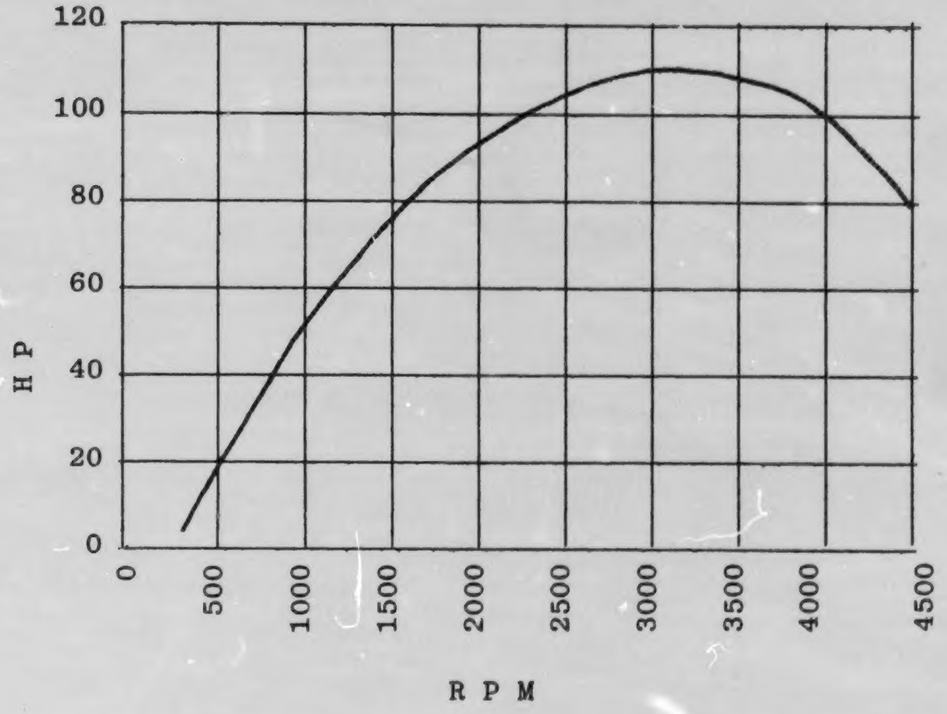
Plot the indicated points on Rectangular Coordinate System and determine the distance between them.

1)	(6, 3)	(8, 3)
2)	(-2, 3)	(4, -5)
3)	(0, 0)	(0, -3)
4)	(-1, -1)	(1, 1)
5)	(1, 4)	(2, 2)
6)	(5, 2)	(-3, 6)
7)	(-2, -3)	(5, 4)
8)	(10, 0)	(-2, 4)
9)	(5, 7)	(1, 4)
10)	(-1, -3)	(-4, -5)

A simple exercise in graphing techniques is to interpret a graph applied to a real world situation.

On the next page, a graph is constructed, representing a "grid". The y axis is in terms of HP → Horsepower. The x axis is expressed as RPM → Revolutions Per Minute.

Try to interpret the grid by determining points through which the curve will pass. The solutions are listed on the following page.



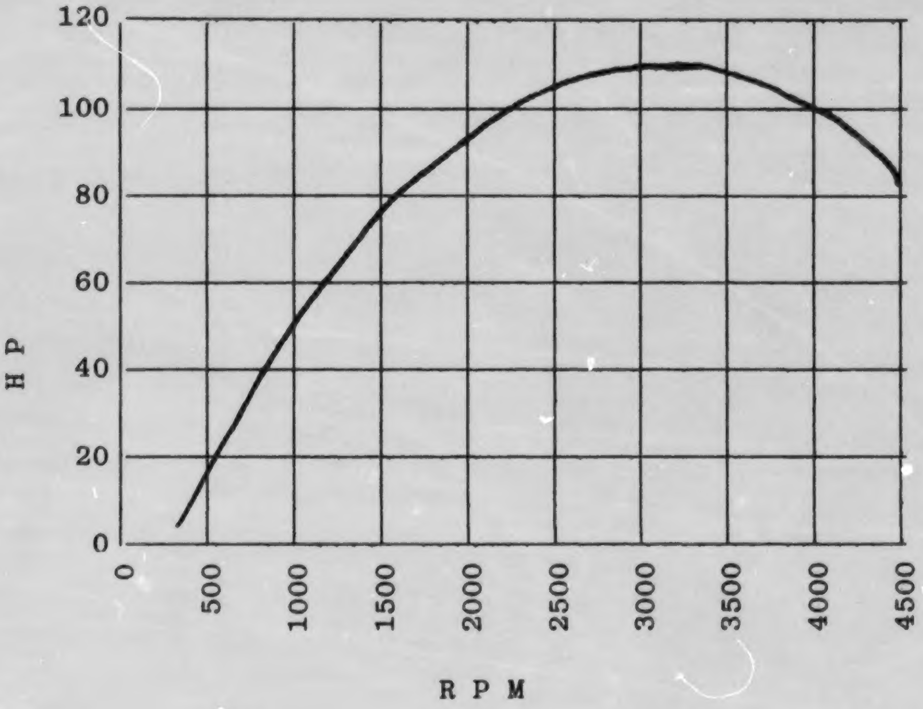
The above graph shows how horsepower varies with speed (in revolutions per minute) for a particular engine. For this engine:

What horsepower (HP) is developed at 500 RPM? ____________

What is the maximum HP the engine can produce? ____________

At what RPM is the maximum HP developed? ____________

As the RPM increases above 3000 RPM does the HP increase, decrease, or stay the same? ____________



The above graph shows how horsepower varies with speed (in revolutions per minute) for a particular engine. For this engine:

What horsepower (HP) is developed at 500 RPM? 20

What is the maximum HP the engine can produce? 110

At what RPM is the maximum HP developed? 3000

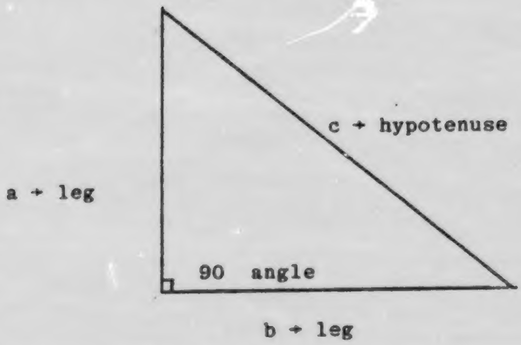
As the RPM increases above 3000 RPM does the HP increase, decrease, or stay the same? Decrease

Distance Between Two Points

The distance between two points on a rectangular coordinate system is easily determined if we first make use of the Pythagorean Theorem.

Pythagorean Theorem → "for any right triangle, the sum of the squares of the legs is equal to the square of the hypotenuse"

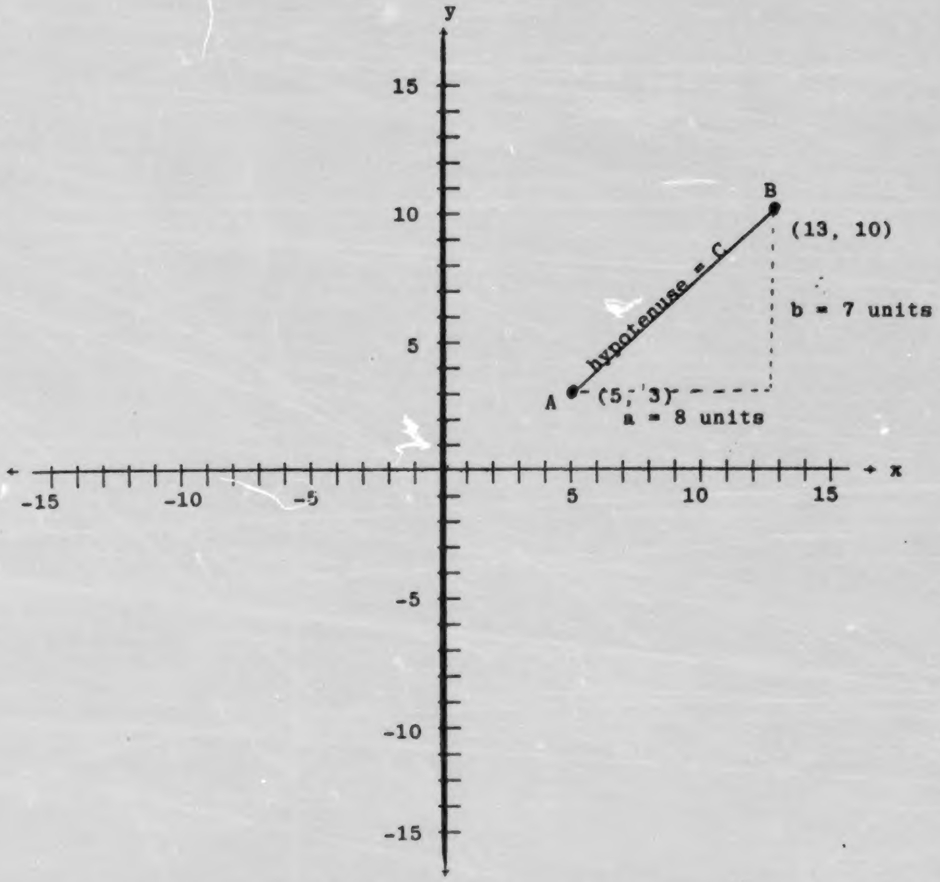
By illustration, the theorem states:



$$\boxed{a^2 + b^2 = c^2}$$

Therefore, if we construct a right triangle between any two points on the rectangular coordinate system, we should be able to find the distance between the two points. For example:

Find the distance between the two points: A = (5, 3) and B = (13, 10).


y
15
10
5
-5
-10
-15
x
-15
-10
-5
5
10
15
B
(13, 10)
hypotenuse = C
b = 7 units
A
(5, 3)
a = 8 units

The horizontal distance, $a \rightarrow x$ axis, is simply the distance of 8 units. The vertical distance, $b \rightarrow y$ axis, is 7 units.

Using the Pythagorean Theorem, we can find the distance between point A and B, which is C, the hypotenuse:

$$
\begin{aligned}
a^2 + b^2 &= c^2 \\
(8)^2 + (7)^2 &= c^2 \\
64 + 49 &= c^2 \\
113 &= c^2 \\
\pm 113 &= c \\
\pm 10.6 &\doteq c
\end{aligned}
$$

Since distance cannot be a negative value in the real world, the correct answer for c is + 10.6 units.

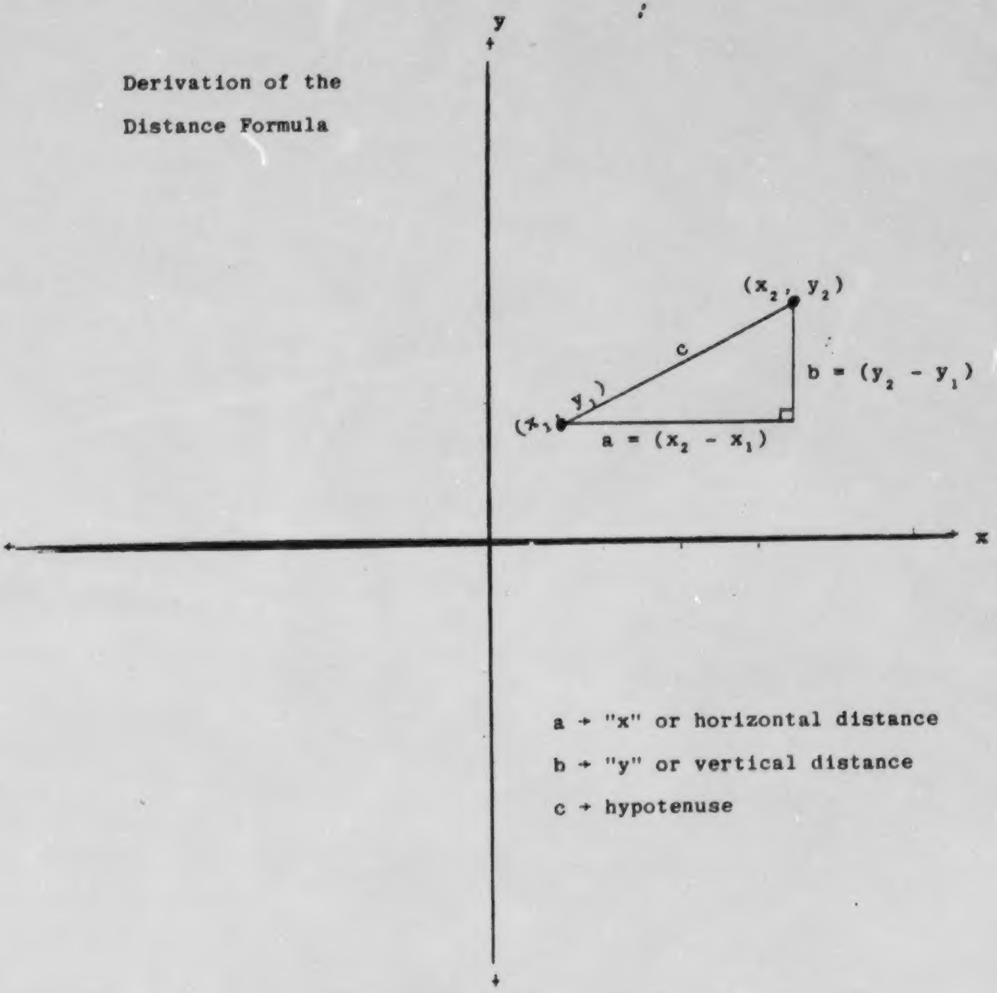
Distance Formula (Between Two Points) - Derived

To derive a distance formula, take two general points: Point 1 = (x_1, y_1) and Point 2 = (x_2, y_2). The numbers 1 and 2 distinguish between the points. And because the numbers (called "subscripts") are written below the line, by convention, they are read as:

$(x_1, y_1) \rightarrow x$ "sub" 1, y "sub" 1

$(x_2, y_2) \rightarrow x$ "sub" 2, y "sub" 2

Now, consider the following graph:


Derivation of the
Distance Formula
y
x
(x₂, y₂)
c
b = (y₂ - y₁)
(x₁, y₁)
a = (x₂ - x₁)
a → "x" or horizontal distance
b → "y" or vertical distance
c → hypotenuse

Substituting these values into the Pythagorean Theorem:

$$c^2 = a^2 + b^2$$

$$c^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2$$

$$c = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \rightarrow \text{DISTANCE FORMULA}$$

Utilizing the Distance Formula

EXAMPLE:

Find the distance between the two points, A = (2, 5) and B = (3, 9)

Solution:

$$A = (2,5) \to (x_1, y_1)$$

$$B = (3,9) \to (x_2, y_2)$$

then, by the distance formula:

$$c = (x_2 - x_1)^2 + (y_2 - y_1)^2$$

$$c = (3 - 2)^2 + (9 - 5)^2$$

$$c = (1)^2 + (4)^2 = 1 + 16 = 17$$

It is important to note that the same answer will result no matter which point is labeled (x_1, y_1) and (x_2, y_2). However, do not mix the subscripts, such as (x_1, y_2).

Linear Equations and Graphs

A linear equation is graphed on a set of axes, x and y; a rectangular coordinate system. In order to graph, our equation takes a different form in terms of a y coordinate, slope, and y intercept.

$y = mx + b$, where

Y → the linear equation $mx + b$

m → slope, or "slant" of the line

b → y intercept; the point on the y axis where the line will cross

Given a rectangular (Cartesian) system and a particular linear equation, the resulting graph will be a straight line based on points x and y. For example:

Given: $y = 2x + 3$

Graph the linear equation.

Solution: Before graphing, we must find points (x, y) which will form the resulting straight line. The student can choose any value for x to substitute and then solve for y.

If $x = 0$; $y = 2x + 3$

$y = 2(0) + 3$ Point 1 = $(0, 3)$

$y = 3$

If $x = 1$; $y = 2x + 3$

$y = 2(1) + 3$ Point 2 = $(1, 5)$

$y = 5$

If $x = -1$; $y = 2x + 3$

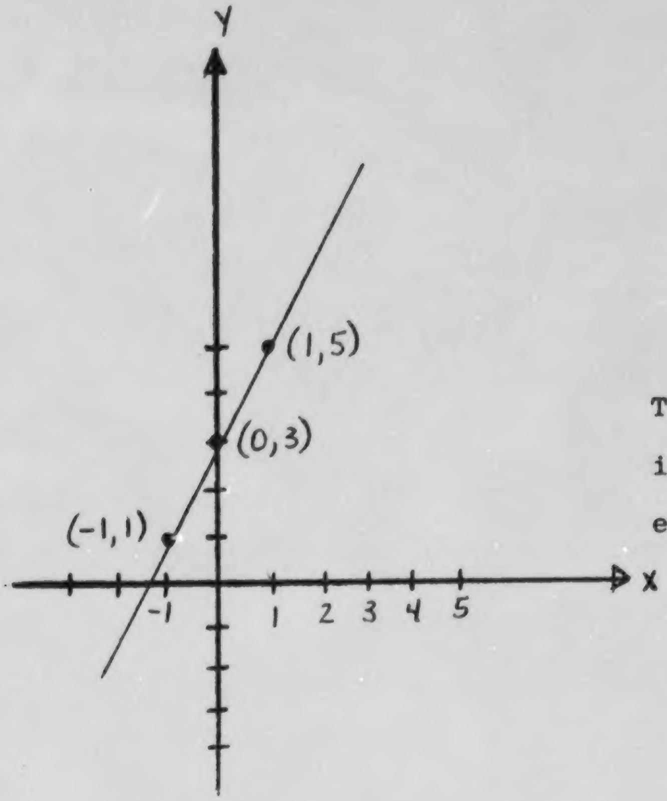
$y = 2(-1) + 3$

Point 3 = $(-1, 1)$

$y = -2 + 3$

$y = 1$

We have found three sets of points (x, y). Plot these points of the x and y axes.



The straight line is labeled with its equation of the line.

Besides plotting points, a linear equation can be graphed by knowing the slope and y intercept.

Slope

The slope of line refers to "how much slant" the line has on a graph. Slope is defined as:

$$\text{SLOPE} \rightarrow m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{\Delta y}{\Delta x}$$

Δ → delta, is another way of saying the "change from one y value to another, or, the difference between two y values".

The slope is also defined as:

$$\text{SLOPE} \rightarrow m = \frac{\text{Rise}}{\text{Run}} = \frac{\uparrow \text{ Vertical Direction}}{\rightarrow \text{ Horizontal Direction}}$$

If a linear equation is given, such as:

$$\boxed{\begin{array}{rcl} y & = & 2x + 3 \\ & & \uparrow \quad\;\; \\ y & = & mx + b \end{array}}$$

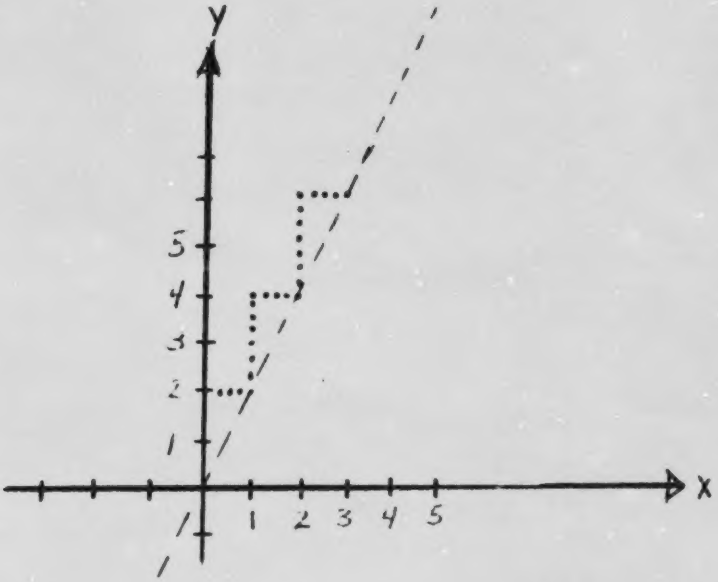
By comparison with the form of a linear equation, m, the slope of a line, is the coefficient of x.

The slope, 2, can be expressed as a ratio:

$$\boxed{\text{Slope} \rightarrow m = 2 = \frac{2 \rightarrow \text{RISE}}{1 \rightarrow \text{RUN}}}$$

After expressing a whole number slope in ratio terms, the slant of all lines having a slope of 2 can be determined.

ROUGH SKETCH
OF SLOPE = 2



Start at the origin. Since the RISE of the slope is 2, go upward 2 units. The Run of the slope is 1, go over 1 unit. Continue the procedure which is depicted on the graph. Next, a broken line is drawn at the slant based on our $\frac{\text{RISE}}{\text{RUN}}$, $\frac{2}{1}$, rough sketch.

The broken line indicates that all linear equations, having a slope of 2, will have this much slant.

Y Intercept

By definition, the y intercept of a linear equation is the point on the y axis where the resulting line will cross, or "intercept".

If a linear equation is given, such as:

$$\boxed{\begin{array}{ccccc} y & = & 2x & + & 3 \\ & & \uparrow & & \uparrow \\ y & = & mx & + & b \end{array}}$$

By comparison with the form of a linear equation, b represents the y intercept, and in the case of $y = 2x + 3$, $b = 3$.

On the graph which a slope of 2 was sketched, we can accurately draw the solid line of $y = 2x + 3$ by first:

1) Draw the solid line parallel to the broken line.

2) Make sure that the solid line crosses, or "intercepts" the y axis at 3.

Therefore, without plotting points, we have constructed an accurate graph of the linear equation, $y = 2x + 3$.

Graphing Straight Lines

Before we discuss the graphing of straight lines, one might ask the question, "how do you tell if an equation is a curve or a line?" This is done by noting the degree of the equation. Straight lines are always 1st degree, often called linear, while most curves are 2nd degree, or higher.

The degree of an equation is determined by the highest power of a single variable. Or, the addition of powers of a product of two or more variables.

For instance, determine the degree of the following:

$x^2 \cdot y = 9$	3rd degree
$x^2 + y^2 = 1$	2nd degree
$x \cdot y = 5$	2nd degree
$x + y = 10$	1st degree, therefore "linear"
$x \cdot y \cdot 7 + x = 10$	3rd degree
$y = 3x + 7$	1st degree, therefore linear

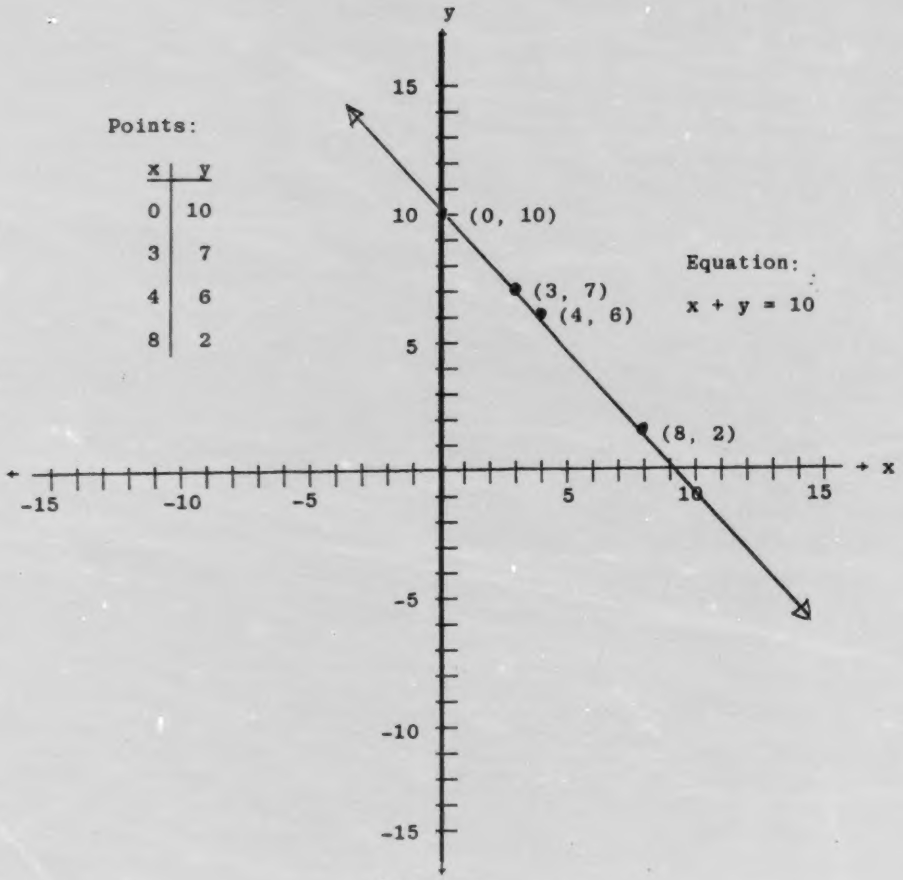
There are two basic methods for graphing linear equations. The first method is to determine two or more points, plot the points on the rectangular coordinate system, and then draw a straight line through the points. A second method, which is often easier and quicker will be discussed in a later chapter.

EXAMPLE:

Graph the following linear equation:

$$x + y = 10$$

In order to graph this equation, we simply complete a chart by randomly choosing x values and then solving for y. We then plot these points and draw a straight line through them.



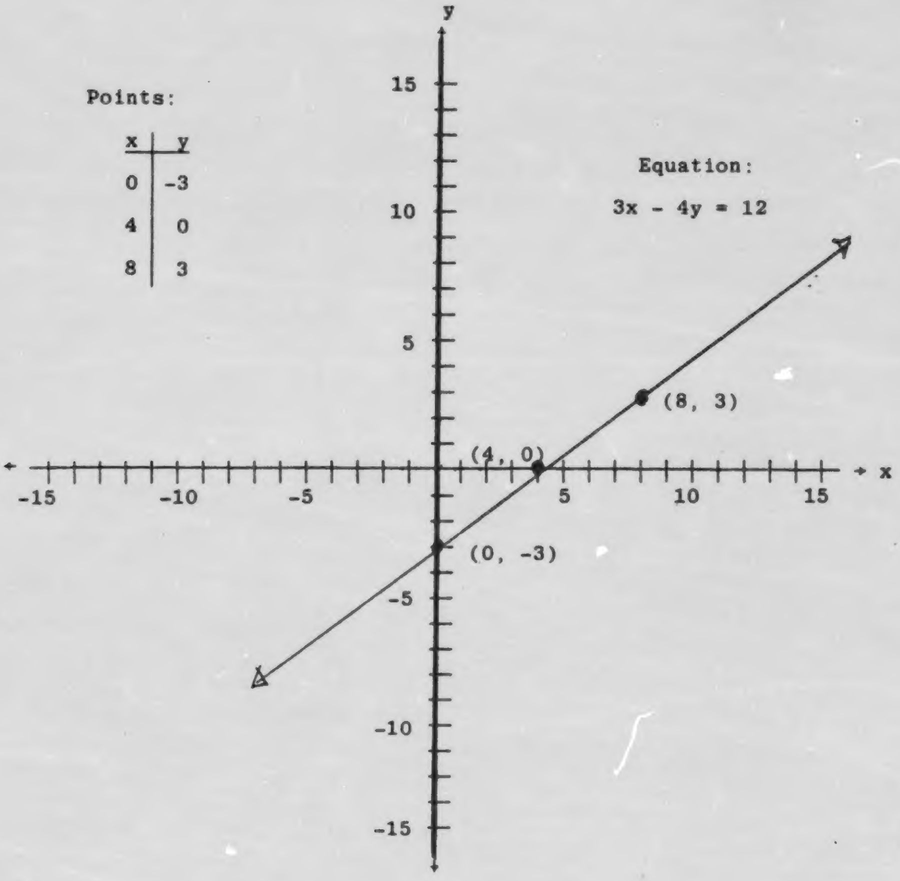
Points:

x	y
0	10
3	7
4	6
8	2

It is important to note that the above line contains all points (x, y) which satisfies the equation $x + y = 10$.

EXAMPLE: Graph the equation:

$$3x - 4y = 12$$



Points:

x	y
0	-3
4	0
8	3

Graph the following lines by choosing x values and determining y. Find at least 3 points.

1) $3x + 2y = 6$

2) $4x - y = 8$

3) $5x + 2y = 10$

4) $x = y$

5) $y = x - 1$

6) $\frac{2}{3}x - \frac{1}{3}y = 1$

7) $x + y = -7$

8) $y - 1 = \frac{2}{3}(x + 1)$

9) $y = -\frac{4}{3}x + 2$

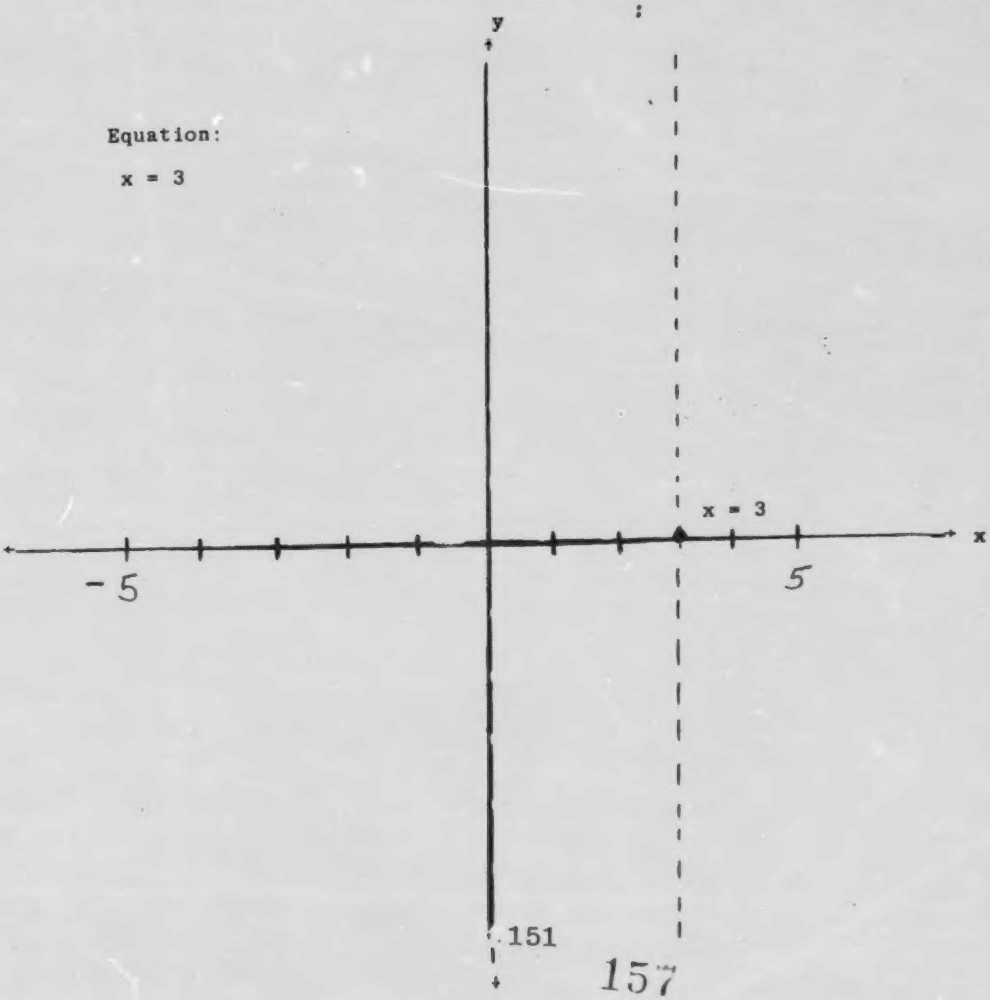
10) $(y - 2) = \frac{1}{4}(x + 3)$

There exist two special cases of graphs of linear equations in which there is only one variable, such as x = A, or y = B, {where A, B are any Real Number}. Graphs of these linear equations are simply vertical or horizontal lines through all points whose abscissas (x values) are equal to A, or whose ordinates (y values) are equal to B, respectfully.

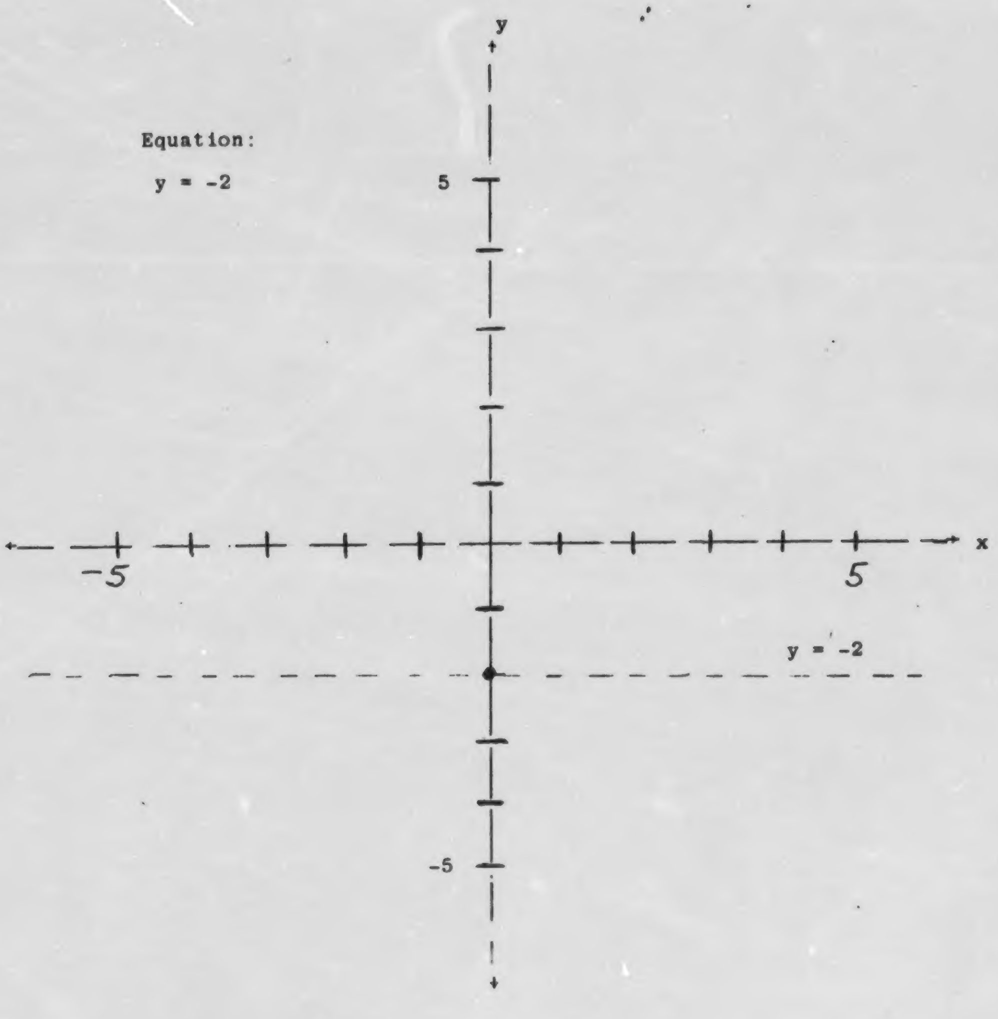
For instance, graph the following two examples:

EXAMPLE 1: x = 3

Since this is a single variable, linear equation, we simply draw a line through all points which have an abscissas value of 3.

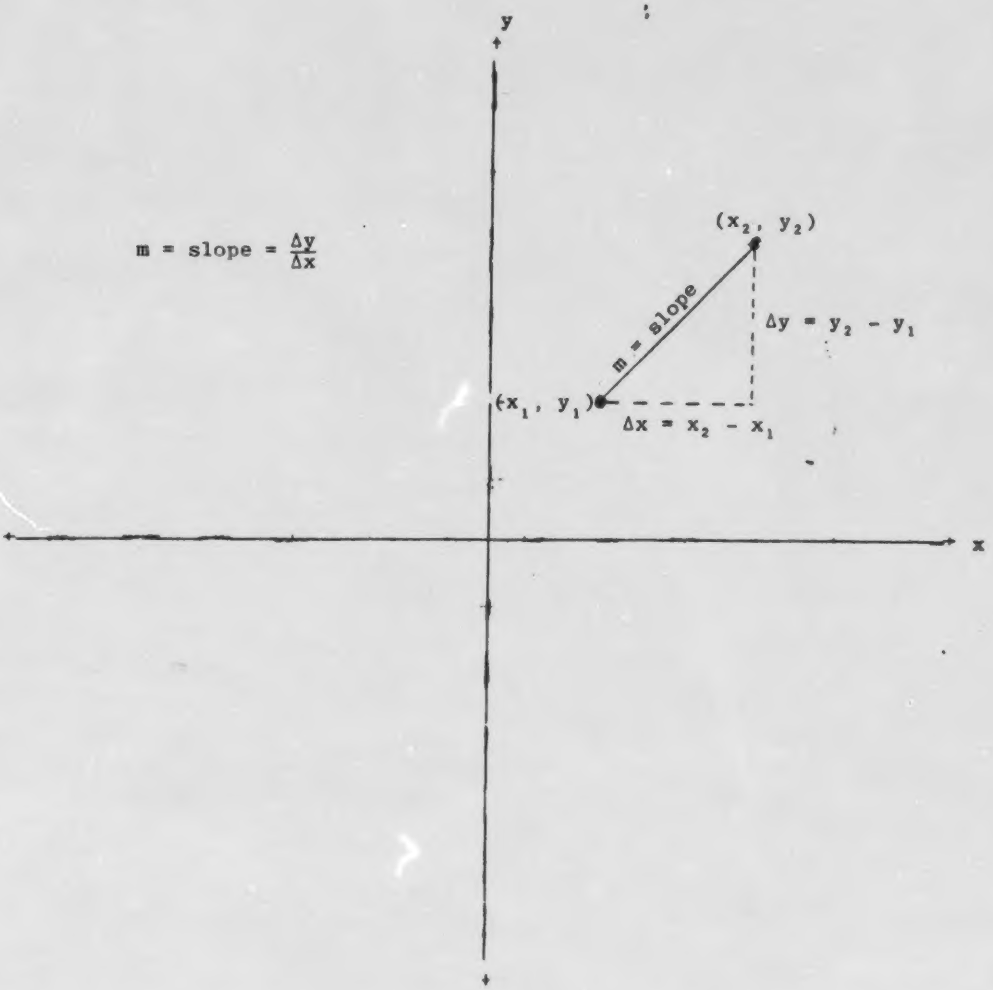


EXAMPLE 2: Graph $y = -2$



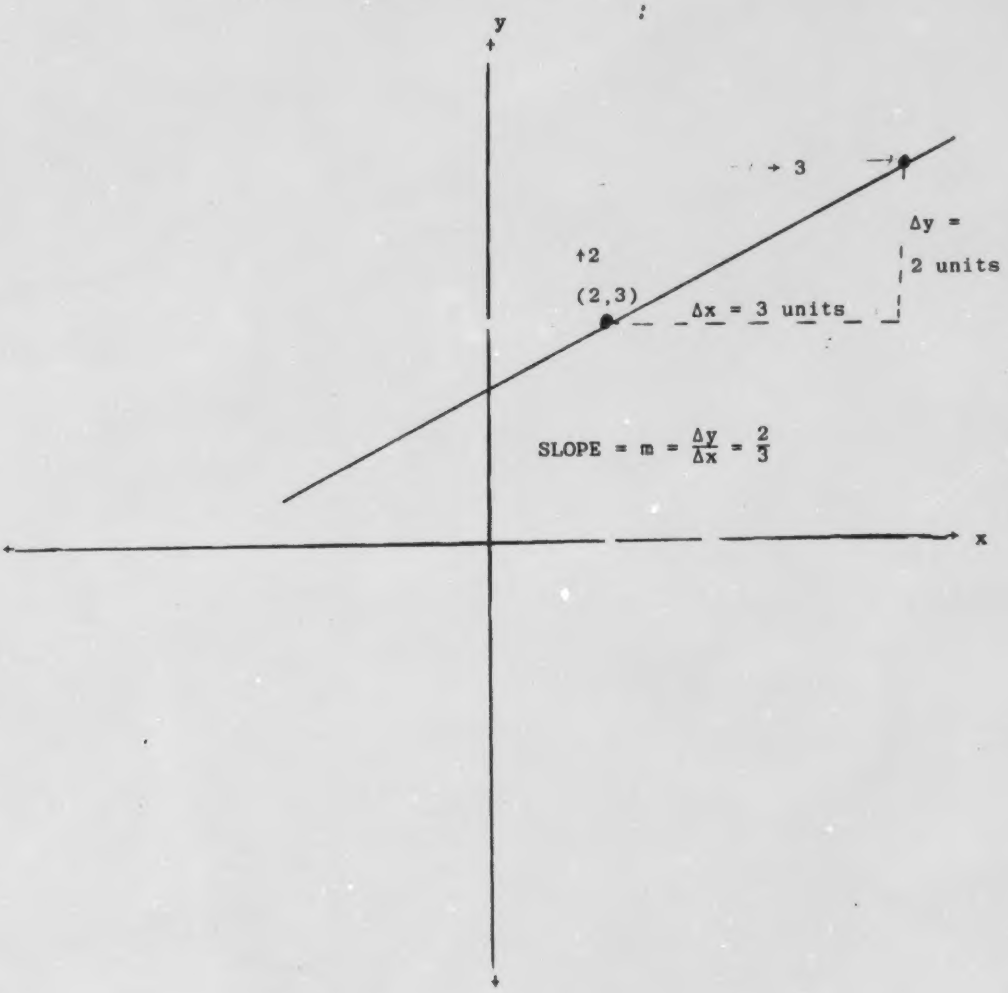
Slope

The slope (denoted by the letter m) of a line between 2 points is equal to the change in y's, divided by the change in x's. Often, slope is defined by the rise divided by the run. Pictorially this would be:


y
x
$m = \text{slope} = \frac{\Delta y}{\Delta x}$
(x_2, y_2)
m = slope
$\Delta y = y_2 - y_1$
(x_1, y_1)
$\Delta x = x_2 - x_1$

*Δ implies change in y and is equal to $y_2 - y_1$.

Knowing the slope and one point on a line, the line can always be constructed. For instance, construct a line on the rectangular coordinate system having m = 2/3 and through the point (2, 3).


y
→ 3
Δy =
2 units
↑2
(2,3)
Δx = 3 units
SLOPE = m = $\frac{\Delta y}{\Delta x} = \frac{2}{3}$
x

Knowing that the slope is the ratio of the Δy divided by the Δx, we may conclude that the slope of a line refers to its steepness, or "slant."

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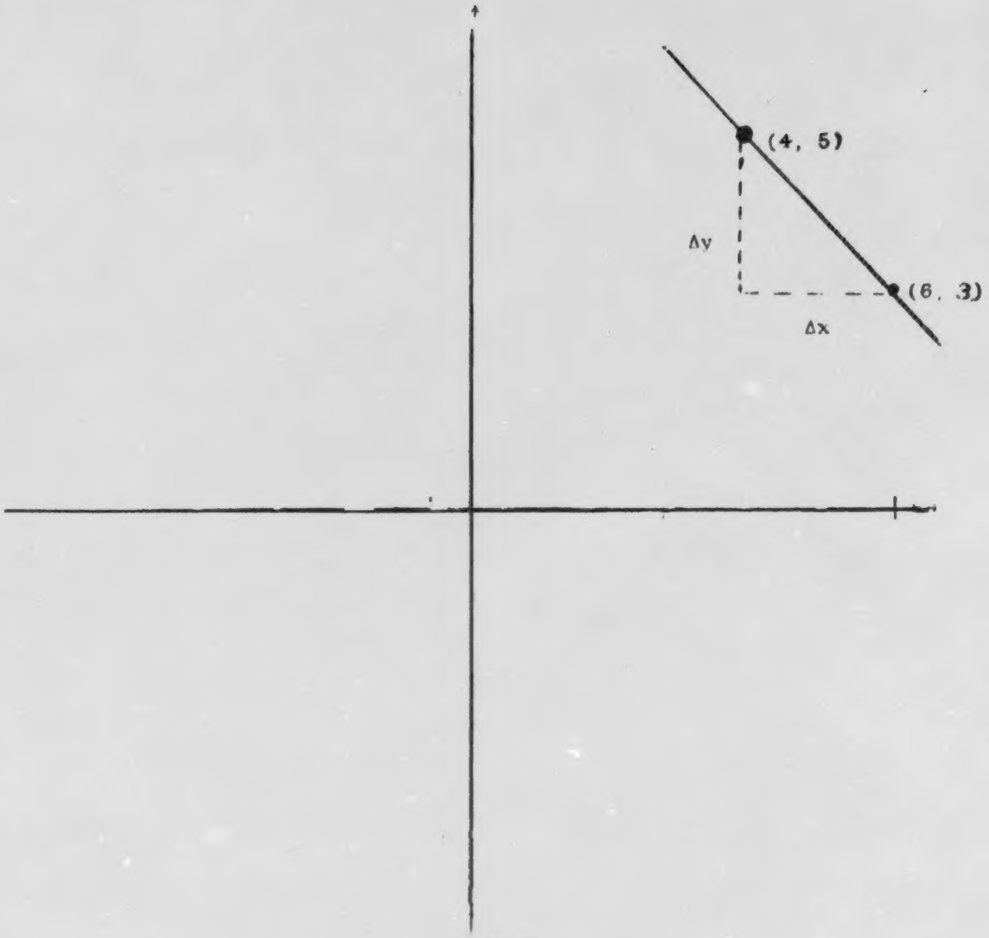
EXAMPLE:

Find the slope for the following and construct the lines showing Δy and Δx.

Given: pt (4, 5) pt (6, 3)

Solution:

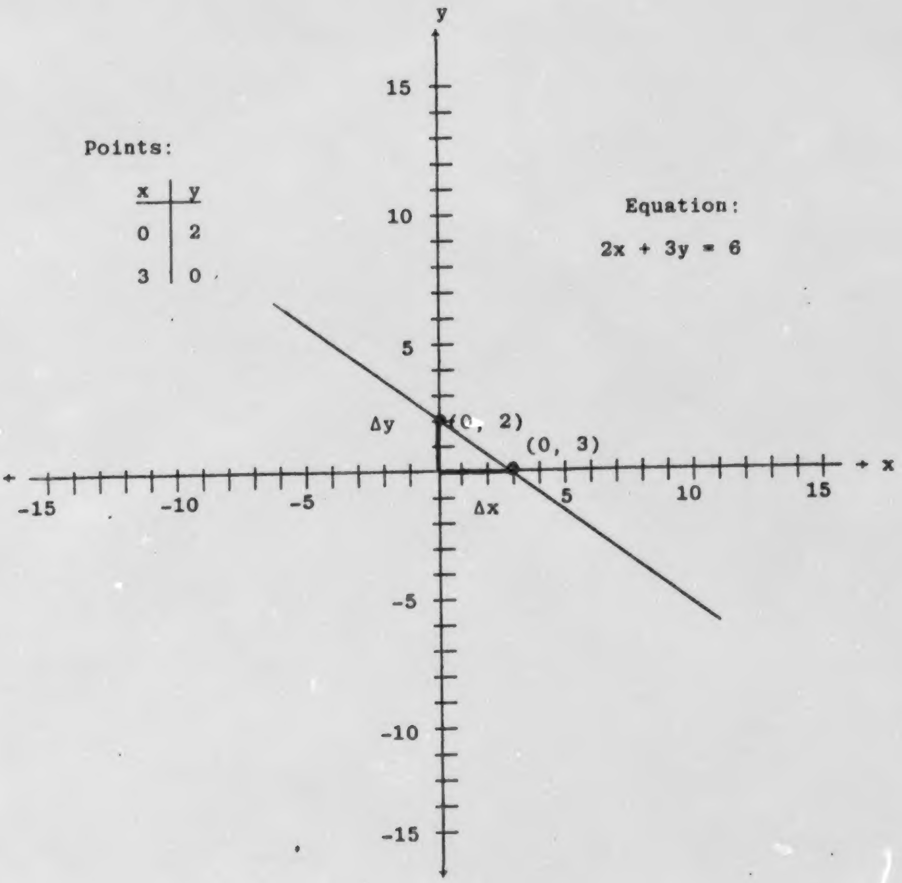
$$m = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{3 - 5}{6 - 4} = \frac{-2}{2} = \frac{-1}{1}$$



EXAMPLE: Find the slope and construct a line for:

Given: $2x + 3y = 6$

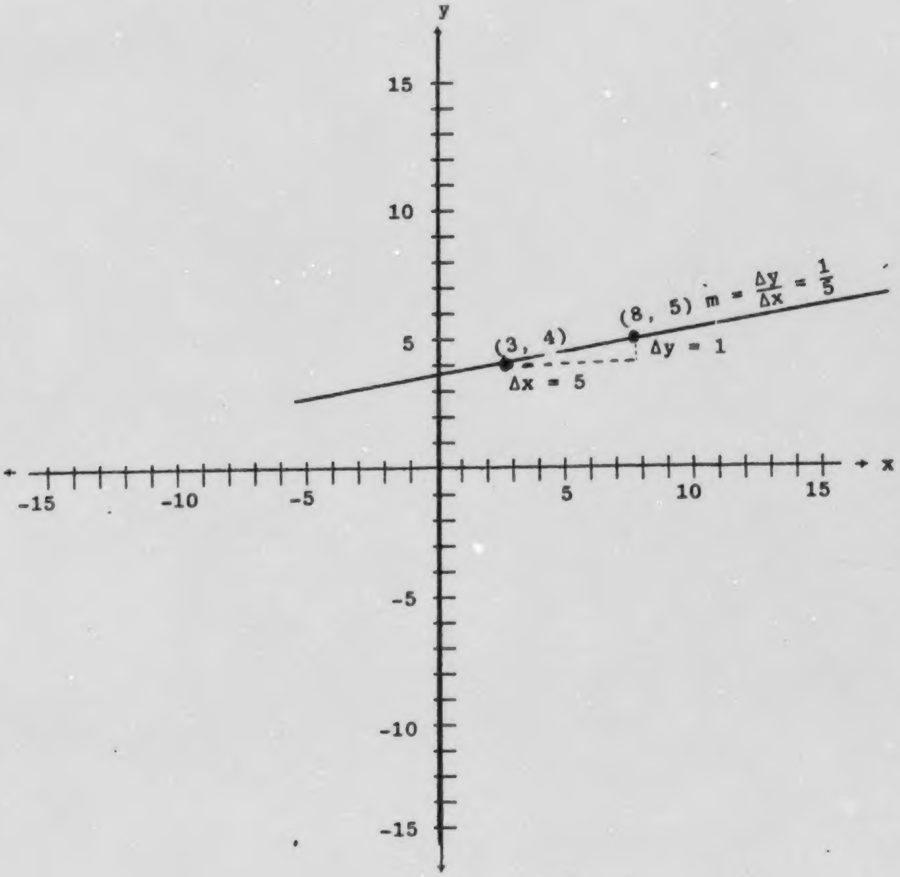
Solution: First we must find some points which satisfy the line $2x + 3y = 6$. This is done by choosing $x = 0$, solving for y; then choose $y = 0$ and solve for x.



EXAMPLE: Find the slope and construct a line for:

Given: pt (3, 4) (8, 5)

Solution: $m = \frac{\Delta y}{\Delta x} = \frac{5 - 4}{8 - 3} = \frac{1}{5}$



It is important that lines which lean to the right {x increases} have positive slope while lines which lean to the left {x decreases} have negative slope. Also, horizontal lines { y = 3, y = 2, etc.} are said to have zero slope since the rise is equal to zero while vertical lines {x = 5, x = 3, etc.} are said to have infinite slope since the denominator is equal to zero.

Graph the Following Lines by Using $y = mx + b$.

Determine the y intercept and the slope of the line.

1) $y + x = 5$

2) $2y - x = 6$

3) $\frac{3}{2}y + x = 6$

4) $y = x - 2$

5) $y - 1 = -\frac{3}{2}(x + 6)$

6) $\frac{3}{4}y - \frac{3}{2}x = 12$

7) $y = 3x$

8) $\frac{x}{2} + \frac{y}{3} = 1$

9) $2y + 3 = x + 6$

10) $x = -\frac{2}{3\,9} + 7$

Determine, if possible, the slope (m) between the following points. Plot the points on the Rectangular Coordinate System. Draw a line through the points and show the slope pictorially on your graph.

1) (5, 2) (6, 4)

2) (2, 3) (-2, -3)

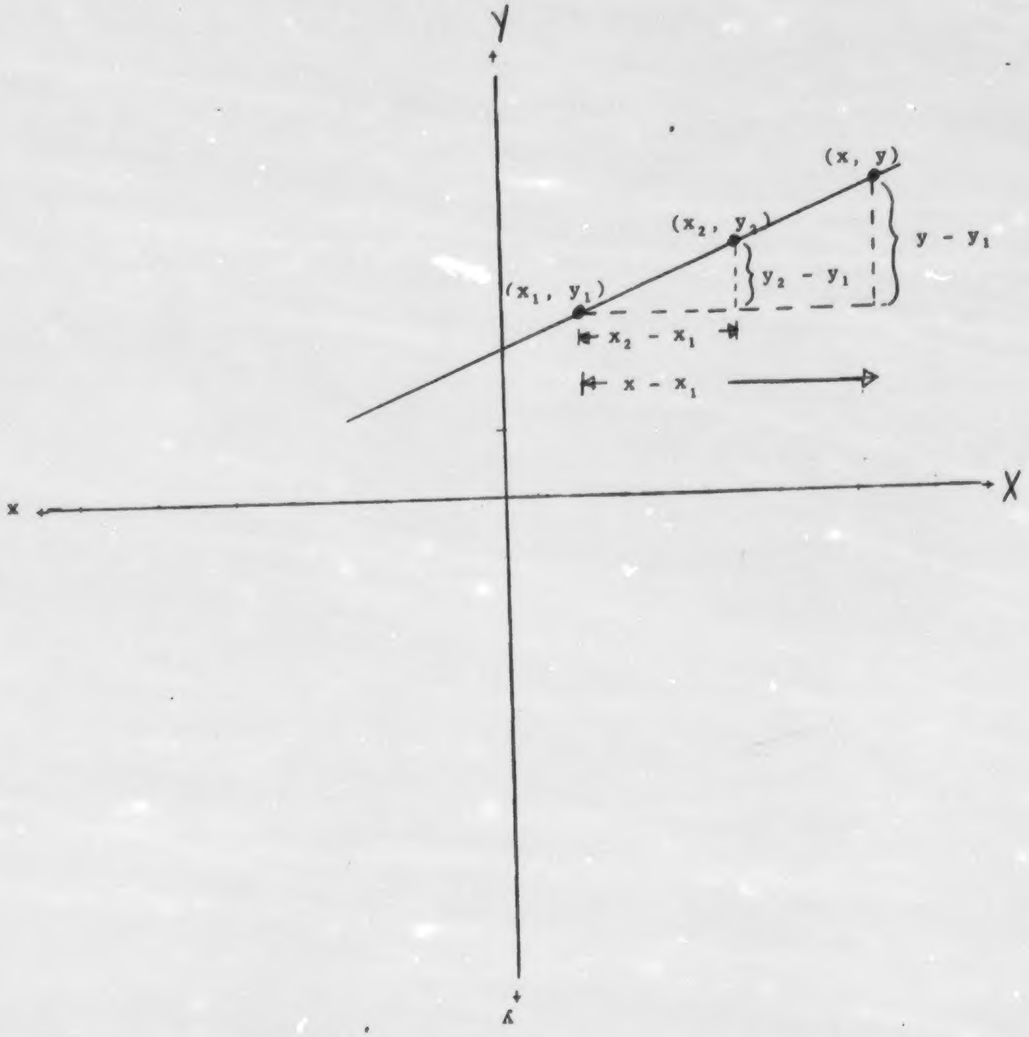
3) (-2, 1) (-4, 2)

4) (6, 3) (9, 3)

5) (5, 2) (5, 9)

Deriving an Equation of a Line

Knowing that the slope, m, is a constant value throughout a given line and knowing two points on the line, we are able to derive an equation which describes every point (x, y) on the line. This is done by choosing two distinct points, (x_1, y_1) and (x_2, y_2) and one arbitrary point (x, y) which is not equal to (x_1, y_1).



Using the fact that the slope, m of a line is a constant value and that the point (x, y) is an arbitrary point on the line, we are able to derive an equation which satisfies all points on the line.

$$m_1 = \frac{y_2 - y_1}{x_2 - x_1} \qquad m_2 = \frac{y - y_1}{x - x_1}$$

$$m_2 = m_1$$

Slope is constant throughout the line, therefore $m_2 = m_1$

$$\frac{y - y_1}{x - x_1} = \frac{y_2 - y_1}{x_2 - x_1}$$

By multiplying both sides of equation by $\frac{(x - x_1)(x_2 - x_1)}{1}$

$$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)$$ → two point formula

or

$$y - y = m(x - x_1)$$ → Point - Slope Formula, where (x_1, y_1) is any point on the line and m is the slope of the line.

Therefore, to derive an equation of a line one simply needs to know 2 points on the line, or one point and the slope of the line.

EXAMPLE: Find the equation of the line through (2,4) and (3, 8).

Solution: Letting $(x\ ,\ y\ ) = (2, 4)$ and $(x_2, y_2) = (3, 8)$ and substituting into the 2 point formula, we would have:

$$y - y = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)$$ → Two Point Formula

$$y - 4 = \frac{8 - 4}{3 - 2} \cdot (x - 2)$$

$$y - 4 = \frac{4}{1}(x - 2)$$

$$y - 4 = 4x - 8$$

$$y - 4x + 4 = 0$$

or multiplying by -1 we have

$$4x - y - 4 = 0$$

EXAMPLE: Find the equation of the line through (-4, -2) having slope = - 2/3.

Solution: $y - y_1 = m(x - x_1)$ → Point - Slope Formula

$$y - (-2) = -\frac{2}{3}\{x - (-4)\}$$

$$y + 2 = -\frac{2}{3}(x + 4)$$

$$3y + 6 = -2(x + 4)$$

$3y + 6 = -2x - 8$ Distributive Property

$$2x + 3y + 14 = 0$$

For consistant results, all lines must be in the form:

$Ax + By + C = 0$ where A, B, C are whole numbers; $A \geq 0$

This form is called standard form.

Find the equation of the line, given the following information. Graph.

1) (1, 2) (5, 7)

2) (6, 2) $m = \frac{2}{3}$

3) (5, 2) (-2, -3)

4) (2, 3) $m = -\frac{4}{5}$

5) (2, 3) $m = 0$

LINEAR EQUATIONS - APPLICATIONS

Linear Equations are utilized in many applications. In this section, we will introduce a few of the most common situations.

Fiscal management of a business involves linear functions which determine demand and revenue, total cost, and break even points. The graphs which illustrate these types of functions are straight lines. Thus, linear equations represent each situation. The familiar terms of slope and y - intercept, take on real-world definitions. Also, the axes x and y have meaning other than abscissa and ordinate.

Consider the form of a linear equation:

$$y = mx + b$$

Related to business, the variables of a linear equation are defined as:

y → Total Cost of producing x units

m → Variable Cost per unit

b → Fixed Cost

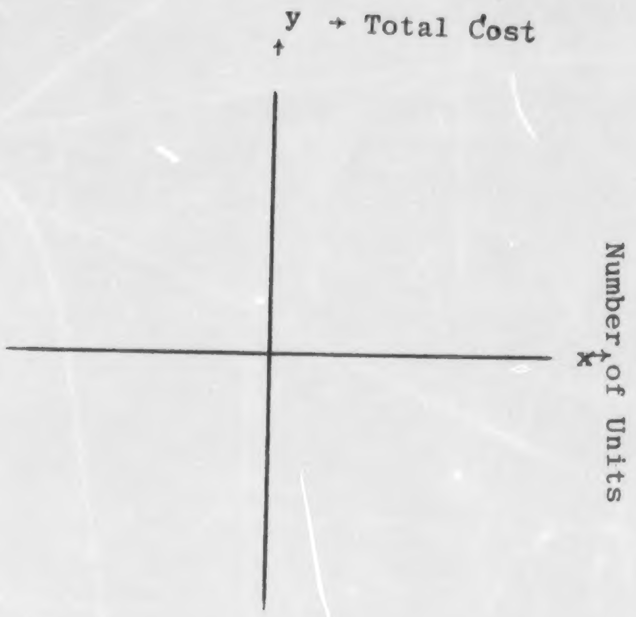
mx → Variable Cost

Further explanations:

Fixed Cost → Any costs which remain constant regardless of the level of production activity.

Variable Cost → Costs that are proportional to production output.

The set of axes x and y is now labeled:



Another important aspect of linear functions in business is the break-even point. By definition, a company will experience a break-even situation when there is neither profit or loss. Specifically, a break-even point is the volume of sales where sales equal production cost.

Referring again to the linear equation:

$$y = mx + b$$

y → Total Cost in dollars

x → Sales in dollars

b → Fixed Cost

With these definitions in mind, the equation becomes known as the cost function, and is represented as:

$$C = mx + F$$

Before attempting to find the break-even point, the Revenue function must be known:

$$R \rightarrow R = (\text{Price})(x, \text{number of units})$$

By setting the revenue function equal to the cost function, $R = C$, we can find the x coordinate of the break-even point. Our notation for this coordinate will be: x_{be}. After finding the x coordinate, substitute the value into the Revenue function. The result will be the y coordinate of the break-even point. Both coordinates are in dollars.

EXAMPLE:

Suppose a company has a cost function: $C = 7x + 1000$. The revenue function is indicated by: $R = 9x$. Find the break-even point and graph the two functions and point on a rectangular coordinate system.

SOLUTION:

First, set $C = R$.

$$\begin{array}{rcl}
7x + 1000 & = & 9x \\
\underline{-9x \qquad\quad} & & \underline{-9x} \\
-2x + 1000 & = & 0 \\
\underline{-\ 1000} & & \underline{-1000} \\
-2x & = & -1000 \\
x & = & 500 = x_{be}
\end{array}$$

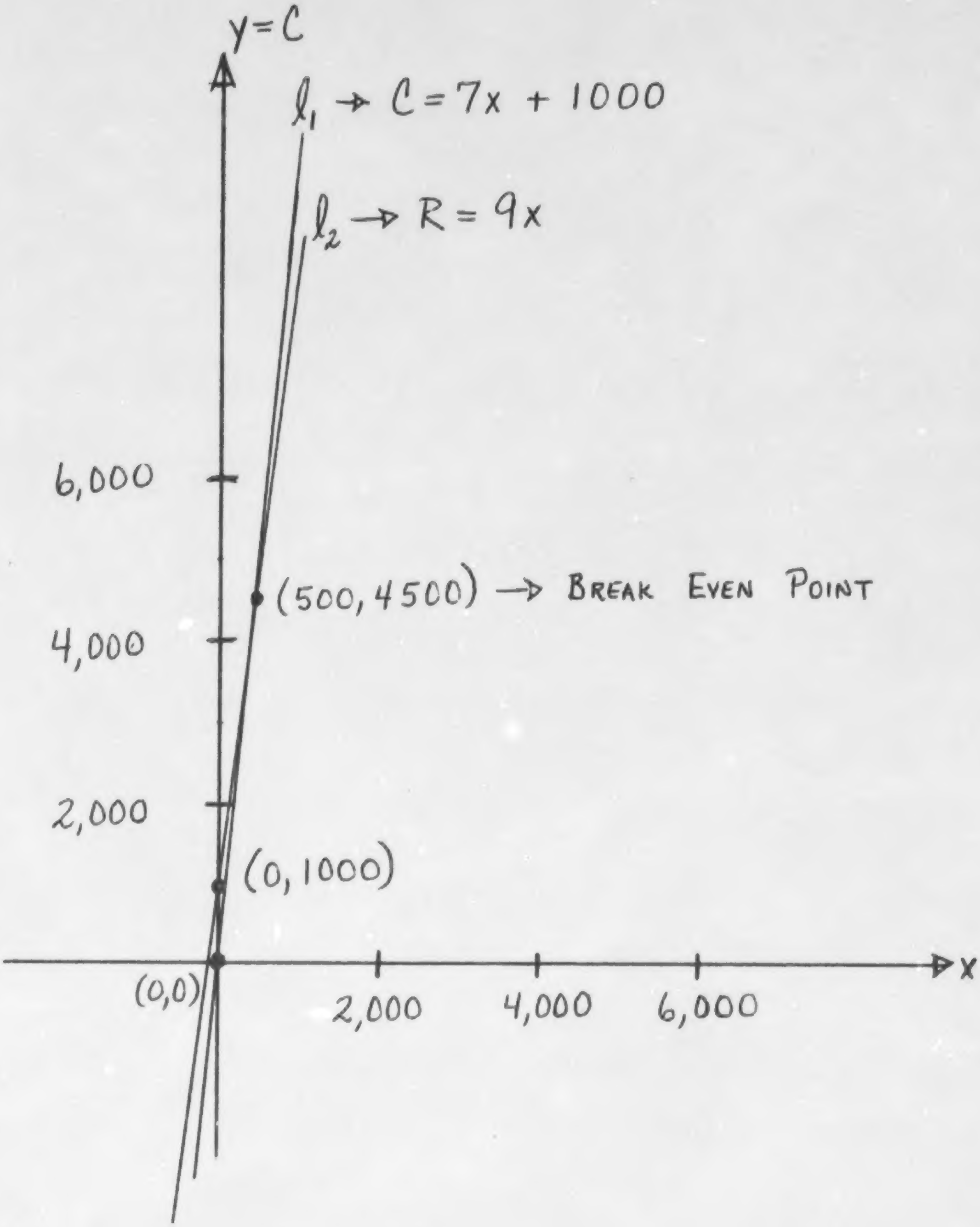
The x coordinate of the break-even point is $x_{be} = 500$. Therefore, the y coordinate is found by:

$$R = 9x$$

$$R = 9x_{be}$$

$$R = 9(500)$$

$$R = 4500 = \text{y coordinate}$$



C = 7x + 1000	
If x = 0	C = 1000
If x = 500	C = 4500

R = 9x	
If x = 0	R = 0
If x = 500	R = 4500

Notice, the intersection of the two lines, Cost and Revenue functions, is also a graphical solution of the break-even point.

EQUATIONS - FIRST DEGREE

(One Variable)

An Introduction

Simple algebraic equations involving one variable with its highest degree one, or first degree, are called linear. The reason is because linear equations graphed on a rectangular coordinate system will produce a straight line.

The mathematical definition of a linear equation is:

$$\boxed{\begin{array}{ll} & ax + b = 0 \\ \text{Where} & a \in \text{Real Numbers} \\ & b \in \text{Real Numbers} \\ & a \neq 0 \end{array}}$$

Remember, a and b are constants from the Real Numbers; they do not represent variables.

Solving linear equations are simple. Once given a particular equation, we solve for the unknown (variable). For example:

Given: $2x + 4 = 0$

Solve for x:

$$\begin{array}{rrcr} 2x & + 4 & = & 0 \\ & -4 & & -4 \\ \hline 2x & & = & -4 \end{array} \quad \text{then: } \frac{2x}{2} = \frac{-4}{2}$$

$$x = -2$$

Another example,

Given: $\frac{x}{x - 3} = \frac{3}{x - 3} + 2$, where $x \in R$.

Solve for x: Before solving for x, we must remove fractions and reduce our equation to its equivalent without fractions.

Step I: $\frac{x}{x - 3} = \frac{3}{x - 3} + \frac{2}{1}$ All terms are in fractional form.

Step II: $\frac{(x - 3)}{1} \cdot \left(\frac{x}{x - 3}\right) = \frac{(x - 3)}{1} \cdot \left(\frac{3}{x - 3}\right) + \frac{(x - 3}{1} \cdot \left(\frac{2}{1}\right)$

(Multiply each term by the LCD over 1.)

Step III: $x = 3 + 2(x - 3)$ Notice our fractions cancel leaving an equivalent, non-fraction equation.

Step IV: $x = 3 + 2x - 6$ Distributive Property.

Now solve for x,

$$\begin{array}{rcl} x & = & 3 + 2x - 6 \\ +6 & & + 6 \\ \hline 6 + x & = & 3 + 2x \end{array}$$

Transposition of constant term 6.

Then,

$$\begin{array}{rcl} 6 + x & = & 3 + 2x \\ - 3 - x & & - 3 - x \\ \hline 3 & = & 1x \end{array}$$

Transposition of constant term 3 and variable x.

Therefore, $3 = x$.

PUZZLES

I. Problem:

The sum of three consecutive even integers is 54. What are the integers?

Solution:

Let the smallest of the three be X. Then the next is $X + 2$ and the third is $X + 4$.

Then, since their sum is 54

$$X + (X + 2) + (X + 4) = 54$$

$$3X + 6 = 54$$

$$3X = 48$$

$$X = 16$$

So the integers are 16, 18, and 20.

II. Problem:

There are 26 coins in a box. There are twice as many nickels as pennies and one-fourth as many dimes as pennies in the box. If there are no other coins in the box what is their total value?

Solution:

Let the number of pennies be X.

Then the number of nickels is $2X$,

and the number of dimes is $\frac{1}{4}X$.

So

$$X + 2X + \frac{1}{4}X = 26$$

or

$$X = 8$$

So the number of pennies is 8, the number of nickels is $2X$ or 16, and the number of dimes is $\frac{1}{4}X$ or 2.

The value of the pennies is $8 \cdot 1 = 8$¢

The value of the nickels is $16 \cdot 5 = 80$¢

The value of the dimes is $2 \cdot 10 = 20$¢

The total value is the sum of these or $1.08.

III. Problem:

A 16 acre property is divided into two parts, one containing 3½ more acres than the other. What are the areas of the two parts?

Solution

Let the area of the smaller part be X. Then the area of the other part is $X + 3\frac{1}{2}$ and their sum is 16.

So

$$X + (X + 3\tfrac{1}{2}) = 16$$

$$2X = 12\tfrac{1}{2}$$

$$X = 6\tfrac{1}{4}$$

$$X + 3\tfrac{1}{2} = 9\ 3/4$$

LINEAR EQUATIONS

Solving for Unknowns in Terms of Variables

If we have a formula, such as A = lw, we can solve for length in terms of Area and width:

$$\frac{A}{w} = \frac{lw}{w}$$

$$\frac{A}{w} = l$$

Width can also be solved in terms of length and area:

$$\frac{A}{l} = \frac{lw}{l}$$

$$\frac{A}{l} = w$$

Other first degree equations in science can be solved in this manner. For instance:

$$^\circ C = {}^\circ K - 273 \qquad \text{Solve for } {}^\circ K$$

$$^\circ C + 273 = {}^\circ K$$

Also,

$V = V_0 + 32t$, Solve for t:

$$\begin{array}{rcl} V & = & V_0 + 32t \\ -V_0 & & -V_0 \\ \hline V - V_0 & = & 32t \end{array}$$

$$\frac{V - V_0}{32} = t$$

Given the equation for perimeter:

$P = 21 + 2w$, Solve for w:

$$\begin{array}{rcl} P & = & 21 + 2\ w \\ -\ 21 & & -21 \\ \hline P - 21 & = & 2w \end{array}$$

$$\frac{P - 21}{2} = w$$

SYSTEMS OF LINEAR EQUATIONS

(Two Variables)

Sometimes, two linear equations will possess the same ordered pair as a solution. When this happens, the two linear equations are referred to as a system.

Given two linear equations in two unknowns, there are three methods used to solve for the ordered pair.

1) Elimination (Addition, Subtraction method)
2) Substitution
3) Graphically

We will give a system, and solve by each method. Remember, each method will result in the same answer (ordered pair).

EXAMPLE:

Given the system $\begin{cases} 2x - y = 4 \\ x + 2y = -3 \end{cases}$

Solve this system for an ordered pair x and y.

METHOD 1 (Elimination)

$$-2\begin{cases} 2x - y = 4 \\ x + 2y = -3 \end{cases} \rightarrow \begin{cases} 2x - y = 4 \\ -2x - 4y = 6 \end{cases}$$

By multiplying one equation in the system, by -2, a new equivalent system results. Notice, the x terms will cancel out if the two equations are added together. Solving for y is quite easy.

$$
\begin{array}{rl}
 & \left\{\begin{array}{rcl} 2x - y & = & 4 \\ -2x - 4y & = & 6 \end{array}\right. \\
\text{add} & \overline{} \\
 & -5y = 10 \\
 & y = -2
\end{array}
$$

Using the same method, we can now solve for x.

$$
2\left\{\begin{array}{rcl} 2x - y & = & 4 \\ x + 2y & = & -3 \end{array}\right. \rightarrow \left\{\begin{array}{rcl} 4x - 2y & = & 8 \\ x + 2y & = & -3 \end{array}\right.
$$

By multiplying the top equation by 2, a new, equivalent system results; the y terms cancel out.

$$
\begin{array}{rl}
 & \left\{\begin{array}{rcl} 4x - 2y & = & 8 \\ x + 2y & = & -3 \end{array}\right. \\
\text{add} & \overline{} \\
 & 5x = 5 \\
 & x = 1
\end{array}
$$

Therefore, the solution (x, y) is (1, -2.)

Solve the Following Systems by Elimination:

1) $\begin{cases} 3x - y = 2 \\ x + y = 2 \end{cases}$

2) $\begin{cases} 2x + 3y = 13 \\ x - y = -1 \end{cases}$

3) $\begin{cases} 3x + 4y = 4 \\ 2x + 5y = 5 \end{cases}$

4) $\begin{cases} 3x + 4y = 17 \\ 2x + 3y = 12 \end{cases}$

5) $\begin{cases} 2x - y = 9 \\ 2y - 4x = -18 \end{cases}$

METHOD 2 (Substitution)

$$\begin{cases} 2x - y = 4 \\ x + 2y = -3 \end{cases}$$

Take one of the two equations and solve for x in terms of y. For instance, choose the top:

$$2x - y = 4$$
$$2x = 4 + y$$
$$x = \frac{4 + y}{2}$$

Now, take this value of x and substitute in the bottom equation of the system; solve for y.

$$x + 2y = -3$$
$$\frac{4 + y}{2} + 2y = -3$$
$$4 + y + 4y = -6$$
$$4 + 5y = -6$$
$$5y = -10$$
$$y = -2$$

To solve for x, use the value of y = -2, and substitute into the "x in terms of y" equation:

$$x = \frac{4 + y}{2} \quad , \quad y = -2$$
$$x = \frac{4 - 2}{2}$$
$$x = \frac{2}{2} = 1$$

Therefore, our solution (x, y) is (1, -2).

Solve the Following Systems by Substitution:

1) $\begin{cases} 3x + 2y = 5 \\ x + y = 2 \end{cases}$

2) $\begin{cases} 2x + 3y = 5 \\ -x + y = 5 \end{cases}$

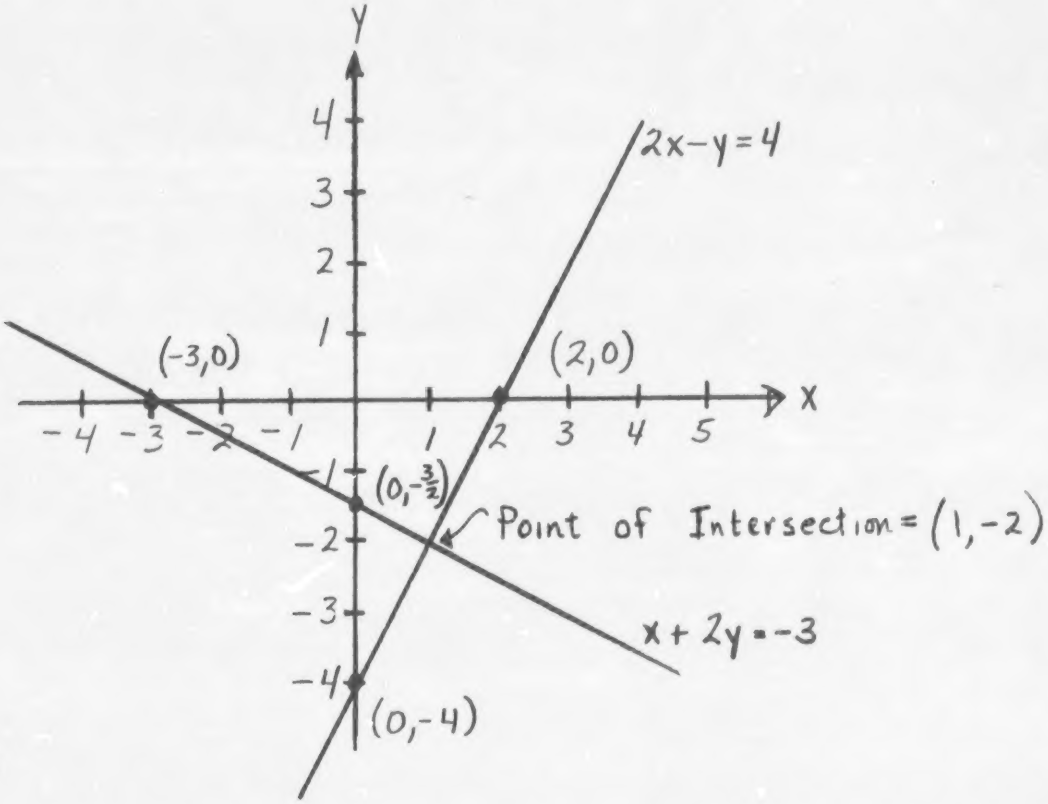
3) $\begin{cases} x + y = 10 \\ 3x + y = 30 \end{cases}$

4) $\begin{cases} 2x - 4y = 4 \\ x + y = 0 \end{cases}$

5) $\begin{cases} 3x + 2y = 2 \\ 6x + 6y = 5 \end{cases}$

METHOD 3 (Graphically)

By plotting points and graphing the line of each equation, if the two lines intersect, the solution will be at the point of intersection.



Not all systems will "share" an ordered pair as a solution. The reasons are beyond the scope of this lesson. However, a simple explanation would involve the graph of the system. If the two straight lines are parallel, meaning they do not intersect, then there is no simultaneous solution to that particular system.

Solve the Following Systems Graphically:

1) $x + y = 0$

$x - y = -4$

2) $3x + 2y = 8$

$y - x = -1$

3) $2y + 3x = 7$

$5x + 2y = 9$

4) $x + y = -7$

$x - y = 1$

5) $3x + 6y = 0$

$5x + 7x = -6$

EQUATIONS - SECOND DEGREE

The degree (highest power) of an equation indicates the number of solutions that will satisfy a given equation.

Therefore, a second - degree (often called quadratic) equation, in standard form, is:

$$\boxed{ax^2 + bx + c = 0}$$

The highest power of x is 2 which means the equation should have two solutions.

There are three basic methods to use in solving this type of equation.

Factoring into Two Binomials

The first method introduced is using the FOIL procedure to factor. A quadratic equation will factor into two binomials. Set each binomial equal to zero and then solve. Why can we set each binomial equal to zero? When the product of two or more expressions is equal to zero, each expression must also be equal to zero.

Example:

Given: $3x^2 + 11x + 6 = 0$

Solve for x

Solution: $3x^2 + 11x + 6 = 0$

$(3x + 2)(x + 3) = 0$

Binomial•Binomial

Set each binomial equal to zero.

Solve for x.

$$\begin{array}{lcl} 3x + 2 = 0 & \text{and} & x + 3 = 0 \\ \quad -2 \; -2 & & \quad -3 \; -3 \\ \hline 3x = -2 & & x = -3 \end{array}$$

Then,

$$\frac{3x}{3} = \frac{-2}{3}$$

$$x = -\frac{2}{3}$$

Answer: $x = -\frac{2}{3}$ and $x = -3$

To check our solutions, take each value for x and substitute into the original equation.

For $x = -\frac{2}{3}$:

$3x^2 + 11x + 6 = 0$ Original equation

$3(-\frac{2}{3})^2 + 11(-\frac{2}{3}) + 6 = 0$ Substitution

$3(\frac{4}{9}) + (-\frac{22}{3}) + 6 = 0$ Multiplication

$\frac{12}{9} - \frac{22}{3} + 6 = 0$

$\frac{12}{9} - \frac{22}{3} + \frac{6}{1} = 0$

$(\frac{9}{9}) \cdot \frac{12}{9} - (\frac{9}{9}) \cdot \frac{22}{3} + (\frac{9}{9}) \cdot \frac{6}{1} = 0$

$\frac{12}{9} - \frac{66}{9} + \frac{54}{9} = 0$

$-\frac{54}{9} + \frac{54}{9} = 0$

$0 \overset{\checkmark}{=} 0$ Checked

For x = -3:

$3x^2 + 11x + 6 = 0$	Original equation
$3(-3)^2 + 11(-3) + 6 = 0$	Substitution
$3(9) + (-33) + 6 = 0$	Multiplication
$27 - 33 + 6 = 0$	
$-6 + 6 = 0$	
$0 \overset{\checkmark}{=} 0$	Checked

Another example of solving a quadratic equation:

Given: $x^2 = -4x - 4$

Solve for x

Solution: First, the equation must be in standard form.

$x^2 = -4x - 4$

Then

$$\begin{array}{rcl} & x^2 = & -4x - 4 \\ \underline{+4x + 4} & & \underline{+4x + 4} \\ +4x + 4 + x^2 = 0 & & \end{array}$$

Proper standard form is:

$x^2 + 4x + 4 = 0$

Factor

$(x + 2)(x + 2) = 0$

Set each factor equal to zero.

$(x + 2) = 0$ and $(x + 2) = 0$

Solve for x

$$\begin{array}{ll} x + 2 = 0 & x + 2 = 0 \\ \underline{\quad -2 \quad -2} & \underline{\quad -2 \quad -2} \\ \quad x = -2 & \quad x = -2 \end{array}$$

In this example, we obtained two solutions, but the values for x are alike. These types of solutions are called double roots. (Note: Root is another way of expressing "answer").

Completing the Square

It is often difficult or impossible to factor quadratic equations in the form:

$$ax^2 + bx + c = 0$$

The second method we introduce to solve quadratic equations is called, completing the square. Though it is used to factor and solve quadratic equations that are difficult, this method can be used to factor and solve easy quadratic equations. The method is lengthy, so the choice is up to you.

Before we discuss the method, it is important to first explain solving quadratic equations where the coefficient of the x term is 1, and the coefficient of b is equal to zero. Basically,

If,

$$ax^2 + bx + c = 0 \quad \text{and } a = 1,\ b = 0$$

then we have,

$$ax^2 + c = 0$$

This type is called a pure quadratic equation.

Example:

Given: $x^2 - 25 = 0$

Solve for x

Solution:

$$x^2 - 25 = 0$$

$$\begin{array}{rcl} x^2 - 25 & = & 0 \\ \underline{\quad + 25} & & \underline{+25} \\ x^2 & = & 25 \end{array}$$

$$x = \pm 25$$

$$x = +5 \text{ and } x = -5$$

Whenever the square root is taken of any number, two solutions, one positive and the other negative, will result.

Another example:

Given: $48 - 3x^2 = 0$

Solve for x :

Solution: $48 - 3x^2 = 0$

$$\begin{array}{rcr}
48 - 3x^2 & = & 0 \\
+ 3x^2 & & + 3x^2 \\
\hline
48 & = & 3x^2 \\
\\
\frac{48}{3} & = & \frac{3x^2}{3} \\
\\
16 & = & x^2 \\
\\
\pm 4 & = & x
\end{array}$$

Now we are ready to investigate the method known as completing the square.

Given: $x^2 + 2x - 3 = 0$

Solve for x: By completing the square

Solution:

$x^2 + 2x - 3 = 0$ Quadratic equation

Step I:

$$\begin{array}{rcl}
x^2 + 2x - 3 & = & 0 \\
+ 3 & & + 3 \\
\hline
x^2 + 2x & = & 3
\end{array}$$

Move the constant to the opposite side of equation.

Step II:

$$x^2 + 2x = 3$$
$$x^2 + 2x + (1)^2 = 3 + (1)^2$$
$$x^2 + 2x + 1 = 3 + 1$$
$$x^2 + 2x + 1 = 4$$

Take one-half the coefficient of x, square the amount, then add to both sides of equation

Step III:

$$x^2 + 2x + 1 = 4$$
$$(x + 1)(x + 1) = 4$$

Factor the left side of equation.

Step IV:

$$\sqrt{(x + 1)(x + 1)} = \pm\sqrt{4}$$

Take the square root of both sides.

Step V:

$$x + 1 = \pm 2$$

Completed square root of both sides.

Step VI:

$$x + 1 = \pm 2$$

Solve for x.

$$x + 1 = +2 \quad \text{and} \quad x + 1 = -2$$

$$\begin{array}{rcr} x + 1 & = & +2 \\ -1 & & -1 \\ \hline x & = & 1 \end{array} \qquad \begin{array}{rcr} x + 1 & = & -2 \\ -1 & & -1 \\ \hline x & = & -3 \end{array}$$

Answer: $x = 1$ and $x = -3$

For another example, we will look again at the quadratic equation mentioned earlier:

Given: $3x^2 - 11x - 4 = 0$

Solve for x using Complete the Square method.

Solution: $3x^2 - 11x - 4 = 0$

Before going further, the coefficient of the x^2 term must always be one. If we divide each term of the equation, both sides, by 3, we have the desired result:

$$\frac{3x^2}{3} \frac{-11x}{3} \frac{-4}{3} = \frac{0}{3}$$

$$x^2 - \frac{11}{3}x - \frac{4}{3} = 0$$

Now proceed with the steps.

Step I:

$$\begin{array}{lcr} x^2 - \frac{11}{3}x - \frac{4}{3} & = & 0 \\ + \frac{4}{3} & & + \frac{4}{3} \\ \hline x^2 - \frac{11}{3}x & = & \frac{4}{3} \end{array}$$

Step II: To take one-half the coefficient of x, multiply the coefficient (which is a fraction), by $(\frac{1}{2})$:

$$\frac{11}{3} \cdot \frac{1}{2} = \frac{11}{6}$$

Square the result.

$$\frac{11}{6}^{\,2} = \frac{121}{36}$$

Add to both sides of equation

$$x^2 - \frac{11}{3}x + \frac{121}{36} = \frac{4}{3} + \frac{121}{36}$$

$$x^2 - \frac{11}{3}x + \frac{121}{36} = \frac{4}{3} \cdot \frac{(12)}{(12)} + \frac{121}{36}$$

Obtaining common denominator on right side.

$$x^2 - \frac{11}{3}x + \frac{121}{36} = \frac{48}{36} + \frac{121}{36}$$

$$x^2 - \frac{11}{3}x + \frac{121}{36} = \frac{169}{36}$$

Step III:

$$\left(x - \frac{11}{6}\right)^2 = \frac{169}{36}$$

Factor left side of equation.

Step IV:

$$\sqrt{\left(x - \frac{11}{6}\right)^2} = \pm\sqrt{\frac{169}{36}}$$

Take square root of both sides.

Step V:

$$x - \frac{11}{6} = \pm\frac{13}{6}$$

Completed square root of both sides.

Step VI: $x - \frac{11}{6} = \pm \frac{13}{6}$ Solve for x

$$x = + \frac{13}{6} + \frac{11}{6} = \frac{24}{6} = 4$$

and

$$x = - \frac{13}{6} + \frac{11}{6} = - \frac{2}{6} = - \frac{1}{3}$$

Quadratic Formula

The third method used in solving quadratic equations is by the Quadratic Formula. This method will solve all second-degree (quadratic) equations. However, it is mainly used to solve those equations which are impossible to factor by the two previous methods discussed.

Deriving the Quadratic Formula

The quadratic formula didn't "just happen". It was derived by "completing the square" on a standard form of a quadratic equation.

Review the steps used in "completing the square". Then, follow those steps as we show how the quadratic formula was derived.

$$ax^2 + bx + c = 0$$

Standard Form Quadratic Equation.

$$\frac{ax^2}{a} + \frac{bx}{a} + \frac{c}{a} = \frac{0}{a}$$

The coefficient of x^2 must always be equal to 1. So, divide each term by the coefficient, a.

$$x^2 + \frac{bx}{a} + \frac{c}{a} = 0$$

Subtract the constant term from both sides.

$$- \frac{c}{a} \quad - \frac{c}{a}$$

$$x^2 + \frac{bx}{a} = - \frac{c}{a}$$

$$x^2 + \frac{bx}{a} + \frac{b}{2a}^{2} = \frac{b}{2a}^{2} - \frac{c}{a}$$

Take ½ of the coefficient of x; square the result; add to both sides.

$$x^2 + \frac{bx}{a} + \frac{b^2}{4a^2} = \frac{b^2}{4a^2} - \frac{c}{a}$$

$$x^2 + \frac{bx}{a} + \frac{b^2}{4a^2} = \frac{b^2}{4a^2} - \frac{c}{a} \cdot \frac{(4a)}{(4a)}$$

Obtaining a common denominator on right side.

$$x^2 + \frac{bx}{a} + \frac{b^2}{4a^2} = \frac{b^2}{4a^2} - \frac{4ac}{4a^2}$$

$$x^2 + \frac{bx}{a} + \frac{b^2}{4a^2} = \frac{b^2 - 4ac}{4a^2}$$

$$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2}$$

Factor left side.

$$\sqrt{\left(x + \frac{b}{2a}\right)^2} = \pm\sqrt{\frac{b^2 - 4ac}{4a^2}}$$

Take square Root of both sides

$$x + \frac{b}{2a} = \pm\sqrt{\frac{b^2 - 4ac}{2a}}$$

$$x = -\frac{b}{2a} \pm\sqrt{\frac{b^2 - 4ac}{2a}}$$

Solving for x

$$x = \frac{-b \pm\sqrt{b^2 - 4ac}}{2a}$$

Therefore x is equal to what is known as the Quadratic Formula.

It is important to note that before using the quadratic formula, the second-degree (quadratic) equations must be in standard form. Also, the coefficient of x^2 should be 1.

Example:

Given: $2x^2 - 9x + 7 = 0$

Solve for x using the Quadratic formula.

Solution: The given quadratic equation is in standard form. Notice:

$2x^2 - 9x + 7 = 0$ Given Equation

$ax^2 + bx + c = 0$ Standard Form

Before using the quadratic formula, the numerical values for a, b, and c must be identified.

$a = 2$

$b = -9$

$c = 7$

Now, solve for x by substituting the values of a, b, and c into the formula:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$ Quadratic Formula stated

$$x = \frac{-(-9) \pm \sqrt{(-9)^2 - 4(2)(7)}}{2(2)}$$ Substitution of values.

$$x = \frac{9 \pm \sqrt{81 - 56}}{4}$$

$$x = \frac{9 \pm \sqrt{25}}{4}$$

$$x = \frac{9 \pm 5}{4}$$

Therefore, the two solutions for x are:

$$x_1 = \frac{9 + 5}{4} = \frac{14}{4} \text{ or } \frac{7}{2}$$

$$x_2 = \frac{9 - 5}{4} = \frac{4}{4} \text{ or } 1$$

Solve by Factoring:

1) $x^2 - 6x + 8 = 0$

2) $x^2 - x - 6 = 0$

3) $x^2 - 16 = 0$

4) $x^2 + 6x - 16 = 0$

5) $9x^2 + 3x - 12 = 0$

6) $4x^2 - 49 = 0$

7) $4x^2 - 3 = -14$

8) $-9x = -14x^2 + 8$

9) $6x^2 - 27x - 15 = 0$

10) $x^2 + x - 20 = 0$

Solve by Completing the Square:

1) $x^2 - 12x + 20 = 0$

2) $x^2 - 8x - 9 = 0$

3) $x^2 - x - 5 = 0$

4) $3x^2 - 4x + 1 = 0$

5) $3n^2 - 8n = -5$

6) $5n^2 + 6n - 8 = 0$

7) $4x^2 - 12x + 9 = 0$

8) $5x^2 + 10x + 1 = 0$

9) $16x^2 - 24x + 9 = 0$

10) $7x^2 - 8x = 2$

Solve by Quadradic Formula:

1) $x^2 + 2x - 8 = 0$

2) $x^2 - 5x + 6 = 0$

3) $2x^2 + 16x - 9 = 0$

4) $9x^2 - 6x - 2 = 0$

5) $2x^2 - x - 2 = 0$

6) $x^2 - 8x + 12 = 0$

7) $x^2 - 20x - 100 = 0$

8) $4x^2 - 7 = -12x$

9) $4x^2 - 5x - 8 = 0$

10) $x^2 + 4x - 4 = 0$

INEQUALITIES

An Introduction

Remember, an algebraic statement containing two expressions which are equal to each other is called an *equation*. But suppose we have two expressions, *not equal* to each other. How would we compare two unequal expressions?

Investigating further, if two expressions are *not equal*, then one expression must have a greater (or lesser) value than the other. Comparison of these type expressions are referred to as *inequalities*.

> An algebraic statement containing two expressions *not equal* to each other, is called a strict inequality.

The basic symbols for strict inequality are:

>	greater than
<	less than

Inequalities - Constant Expressions

If we are confronted with a statement of inequality, there is a question concerning the truth or falsity of the statement.

A statement involving constant expressions only (i.e., containing no variables), can be proved or disproved by observation.

EXAMPLE:

$5 > 2$, read "five is greater than two", is a true statement.

EXAMPLE:

$5 < 2$, read "five is less than two", is not a true statement.

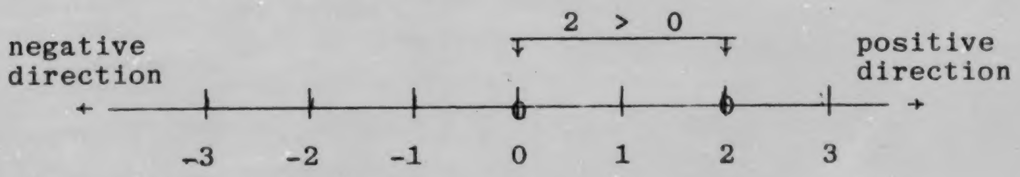
EXAMPLE:

$4 + 5 \cdot 2 > 3 \cdot \frac{10}{2}$ Calculate. Then determine the truth or falsity.

$4 + 10 > \frac{30}{2}$

$14 > 15$ True

It is often helpful to recall the number line when comparing two numbers. As we progress to the right on a number line, the values become larger. Progressing to the left, numbers become smaller. This helps to avoid confusion when dealing with negative numbers.



Inequalities - Variable Expressions

Inequalities involving variable expressions depend on the values of each variable. When given such an inequality, the expression must be solved for the variable, and then the solution placed into the original inequality to determine truth or falsity. Solving for variables (unknowns) of inequalities is referred to as solutions of the inequality.

In order to solve an inequality of variable expression, the method used is the same as solving for unknowns in equations. For example:

Solve for x: $2x - 3 = 11$

$$\begin{array}{rcl} 2x - 3 & = & 11 \\ \underline{\quad + 3} & \underline{\quad} & \underline{+ 3} \\ 2x & = & 14 \quad \text{then } x = 7 \end{array}$$

An inequality would be of a type:

$$2x - 3 < 11$$

By observation, we cannot tell what values of x will make this a true statement. To find these values of x, we apply the same rules for solving equations. Therefore,

$$\begin{array}{rcl} 2x - 3 & < & 11 \\ \underline{\quad + 3} & \underline{\quad} & \underline{+ 3} \\ 2x & < & 14 \\ x & < & 7 \end{array}$$

If $x < 7$, then choose a value, say $x = 6$. Substitute $x = 6$ into the original inequality.

$$2x - 3 \overset{?}{<} 11$$

$$2(6) - 3 \overset{?}{<} 11$$

$$12 - 3 \overset{?}{<} 11$$

$$9 < 11$$

It can be seen that after substituting a value "less than 7", such as $x = 6$, the inequality is true. And therefore, the original inequality statement of $2x - 3 < 11$ is true for values $x < 7$.

Consider the equation:

$$-2y = 16$$

$$\frac{-2y}{-2} = \frac{16}{-2}$$

$$y = -8$$

The solution was easy. However, if the example had been an inequality statement, an important rule would apply. For instance, study the next example closely:

$$-2y < 16$$

$$\frac{-2y}{-2} < \frac{16}{-2}$$

$$y > -8$$

Each time both sides of an inequality expression is divided by a negative number, the inequality symbol is reversed.

After obtaining $y > -8$, we choose a value "greater than -8" and substitute into the original inequality expression. A value, greater than -8, could be -7. Remember, progressing right on a number line will produce larger numbers. Choosing $y = -7$:

$$-2y \overset{?}{<} 16$$

$$-2(-7) \overset{?}{<} 16$$

$$14 \overset{?}{<} 16$$

Choosing -7 as a value greater than -8

Expression is true.

If we combine the equal symbol, =, with the inequality symbols < or >, we form the following:

$\leq$ "less than or equal to"

$\geq$ "greater than or equal to"

A statement of two expressions, compared as a strict inequality (< or > only), could be determined as false. However, the symbols $\leq$ or $\geq$ can turn the statement into a truth. For example:

$5 > 5$ False Statement

$5 \geq 5$ True Statement

But, considering other examples:

$4 > 5$ False Statement

$4 \geq 5$ Also, False Statement

And,

$6 > 5$ True Statement

$6 \geq 5$ True Statement

Symbols $\leq$ or $\geq$ can sometimes change a statement from false to true.

It is important to remember that a strict inequality is true only when the inequality symbol $>$ is pointing toward the smaller number. An inequality involving the "equal to" sign ($\leq$ or $\geq$) is true only when the symbol is pointing toward the smaller number, or when the two numbers (or expressions) are equal.

Solve and graph the following inequalities:

1) $x + 2 > 5$

2) $-3x \leq 6$

3) $-2(x + 3) \geq -4$

4) $3x + 7 \geq 1 + 2x$

5) $\frac{4x}{3} + 2 \geq -6$

LINEAR INEQUALITIES

Before we discuss linear inequalities it is important to remember that a linear equation containing two variables, x and y, can be solved graphically as a collection of points forming a straight line.

The solution of a linear inequality can be represented by an area in a rectangular coordinate system bounded on one side by a straight line.

To graph a linear inequality we first write the inequality as a linear equation. This is accomplished by replacing the inequality sign with an equal sign. Next we graph the linear equation. If we are dealing with a strict inequality we represent the linear equation as a dotted line in our coordinate system. If the inequality is not strict then we represent the line in the normal manner. To finish our solution we must determine the correct area representing the inequality. To do this we select a point on either side of the line (being careful not to select a point that lies on the line) and see whether the coordinates satisfy the inequality. If the inequality is satisfied then the point lies on the correct side of the line. If the inequality is not satisfied then the correct area is on the other side from the point selected.

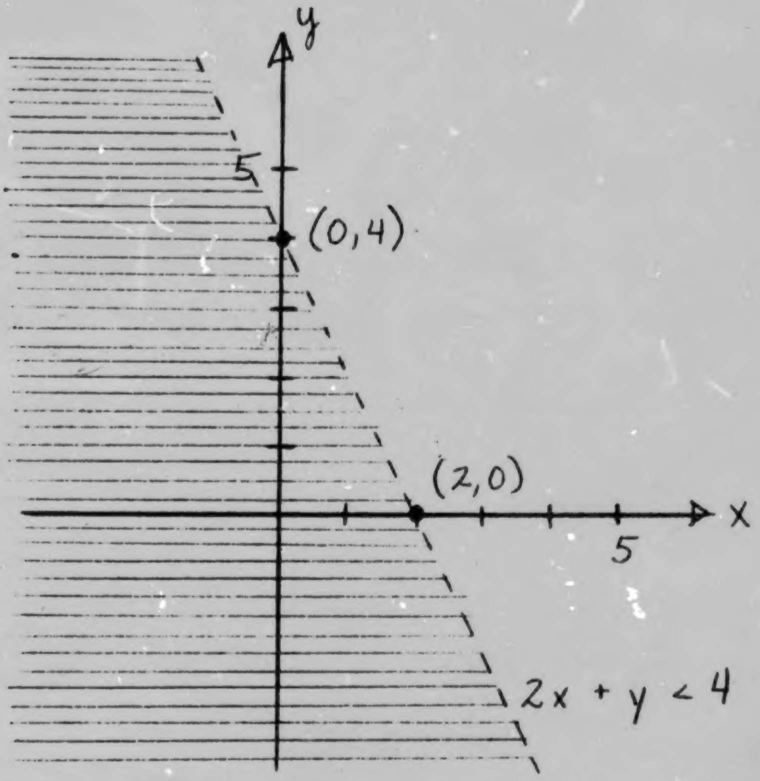
EXAMPLE:

Solve: $2x + y < 4$

First we write the inequality as an equation:

$$2x + y = 4$$

We graph the equation in a rectangular coordinate system.



Notice we have graphed the equation as a dotted line because we are dealing with a strict inequality. Points lying on the line are not solutions to the inequality, thus we represent the line in this fashion. Next, we select a point to test the inequality. In most cases it is convenient to choose the origin as our test point. The coordinates of the origin are (0, 0). Substituting these coordinates into the inequality:

$$2(0) + 0 < 4$$

or

$$0 + 0 < 4$$

Thus these coordinates satisfy the inequality so the origin lies in the correct area representing the solution. We denote this by shading in this area.

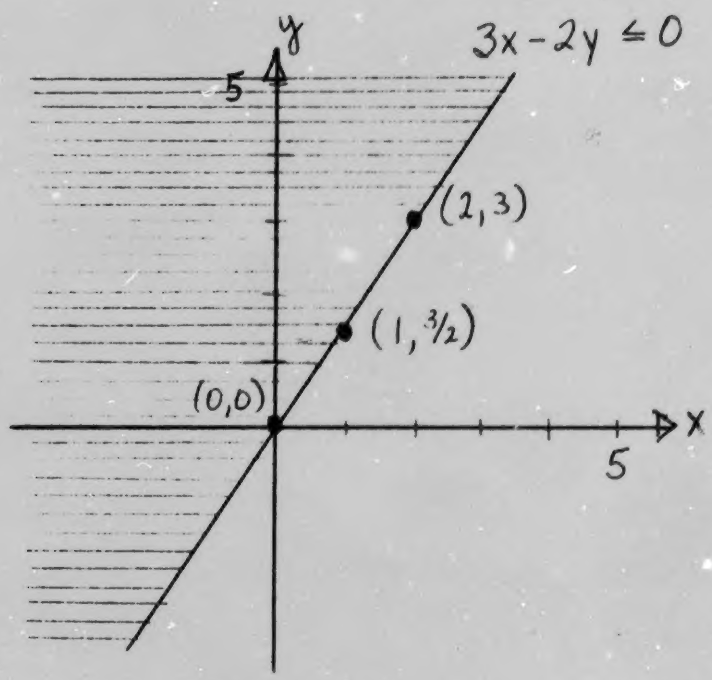
EXAMPLE:

Solve: $3x - 2y \leq 0$

Writing the inequality as an equation we get:

$$3x - 2y = 0$$

Next we graph the equation:



Notice that we are not dealing with a strict inequality so we have graphed our straight line in the normal fashion. Points lying on this line are solutions to the inequality. Now we select a test point (notice we cannot select the origin because it lies on the line). Choosing the point with coordinates (2, 1), we substitute these coordinates into the inequality.

$$3(2) - 2(1) \leq 0$$

$$6 - 2 \leq 0$$

$$4 \leq 0$$

Since the inequality is false then we shade the area opposite the side our test point lies in as our solution.

Graph the following linear inequalities:

1) $x + y \geq 2$

2) $3x + 4y < 8$

3) $-2y \geq 8x - 10$

4) $x \geq 2$

5) $y \geq -3$

SYSTEMS OF LINEAR INEQUALITIES

We have learned that the solution of a linear inequality can be represented by a certain restricted area in a rectangular coordinate system.

The solution of a system of linear inequalities can be represented as the intersection of the areas representing the solution of each individual inequality.

To find this intersection, we first graph each inequality on the same coordinate system and then observe the intersection of the areas representing each individual solution. The resulting area may be completely enclosed or restricted in one or two directions. If there is no intersection between the individual areas then there is no solution to the system.

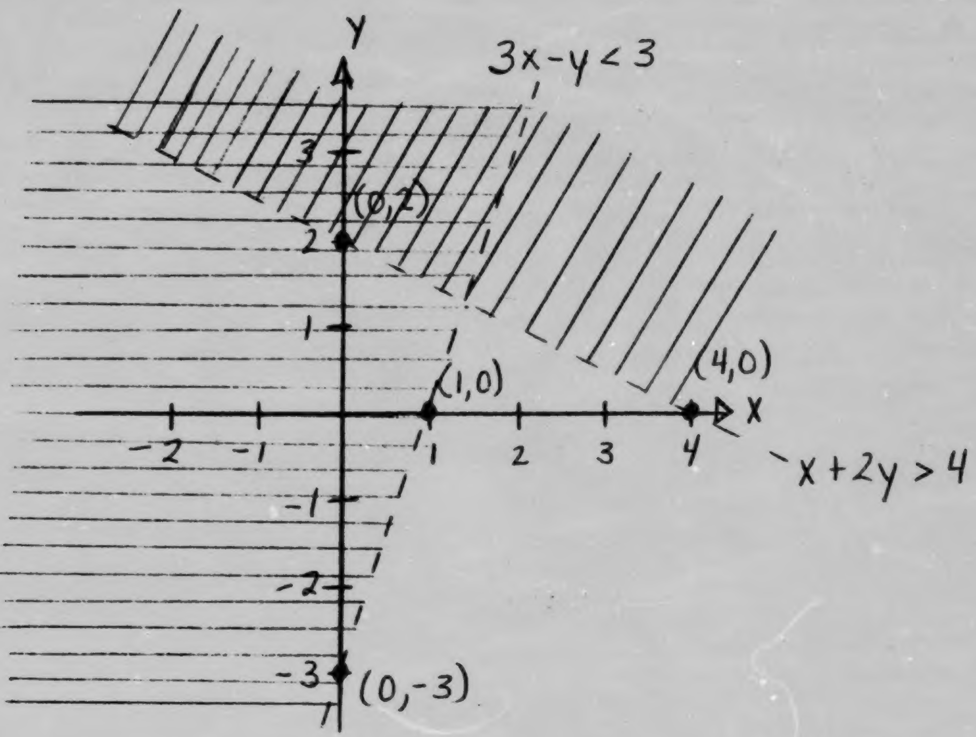
EXAMPLE:

$$x + 2y > 4 \quad \text{and} \quad 3x - y < 3$$

Writing the inequalities as equations, we have:

$$x + 2y = 4 \quad \text{and} \quad 3x - y = 3$$

Next we graph each inequality on the same coordinate system.



We can use (0, 0) as our test point for both inequalities. From the first inequality, we get

$$(0) + 2\,(0) > 4$$

$$0 > 4 \quad , \quad \text{which is false.}$$

For the second inequality, we get

$$3\,(0) - 0 < 3$$

$$0 < 3 \quad , \quad \text{which is true.}$$

After shading in the appropriate areas, the intersection is represented by the double shaded area. The double shaded area indicates the solution for our system of inequalities.

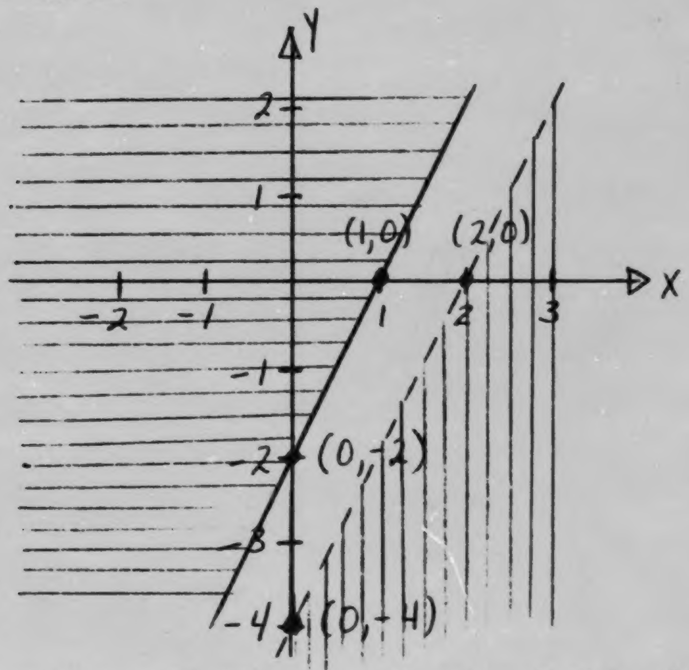
EXAMPLE:

$$4x - 2y > 8 \quad \text{and} \quad 2x - y \le 2$$

Writing the inequalities as equations:

$$4x - 2y = 8 \quad \text{and} \quad 2x - y = 2$$

When graphing, notice these two equations represent parallel lines.



We can again use (0, 0) as our test point for both inequalities. For the first, we get

$$4(0) - 2(0) > 8$$

$$0 > 8 \quad , \quad \text{which is false.}$$

For the second, we get

$$2(0) - 0 < 2$$

$$0 < 2 \quad , \quad \text{which is true.}$$

After shading the appropriate areas, we see there is no intersection, therefore this system has no solution.

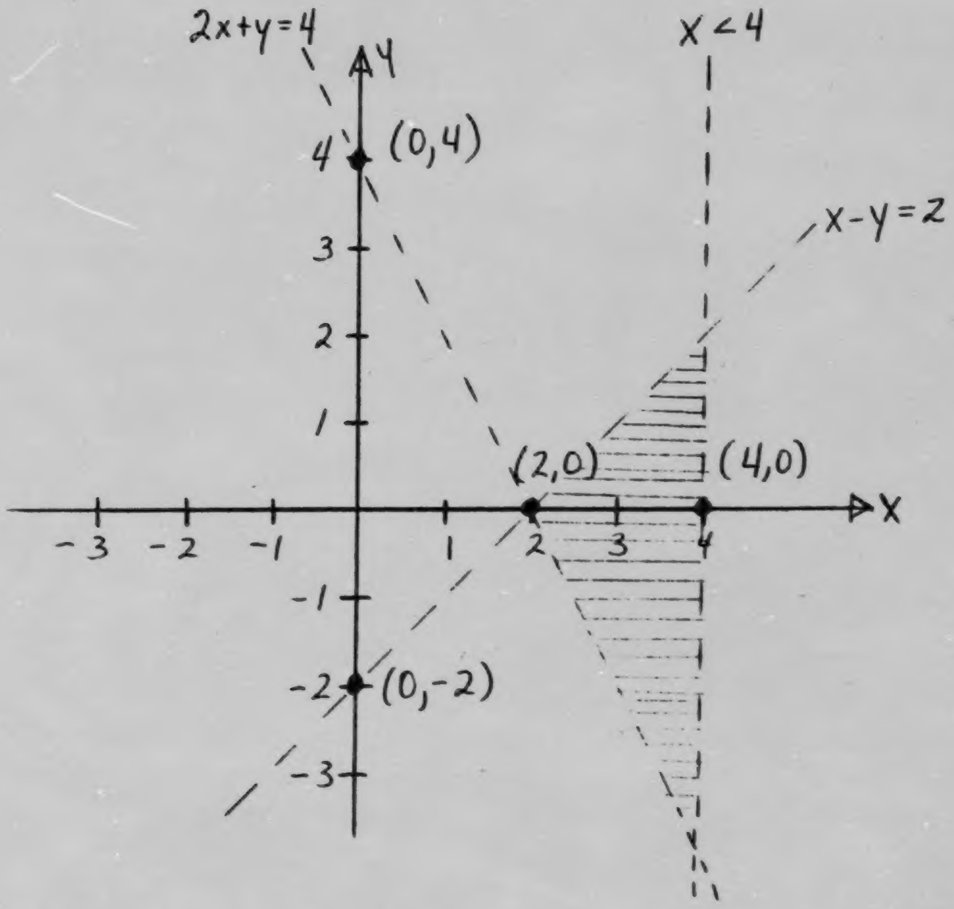
EXAMPLE:

$$x - y > 2; \quad 2x + y > 4; \quad x < 4$$

Writing the inequalities as equations, we get

$$x - y = 2; \quad 2x + y = 4; \quad x = 4$$

Now we graph each equation



Using the origin (0, 0) as our test point for all three inequalities, we find the enclosed shaded area to be our solution for this system.

Graph the following system of linear inequalities:

1) $\begin{cases} x + 2y \geq 5 \\ y - x \leq 0 \end{cases}$

2) $\begin{cases} 2x + 3y > 6 \\ x + 4y > 8 \end{cases}$

3) $\begin{cases} y > 4 \\ x < 2 \end{cases}$

4) $\begin{cases} x - y \geq 7 \\ x + y \leq 7 \end{cases}$

5) $\begin{cases} 2x + 3y \leq 6 \\ 3x + 4y \geq 12 \end{cases}$

QUADRATIC INEQUALITIES

Solving quadratic inequalities can be accomplished by either algebraic or graphical methods.

Algebraic Method

To solve algebraically, we first transpose all terms to one side of the inequality. Next, factor the quadratic expression, if possible. By properties of signed numbers, we can now find solutions to the inequality, if they exist.

EXAMPLE: $x^2 + 3 > -4x$

$x^2 + 4x + 3 > 0$ transposition

$(x + 1)(x + 3) > 0$ factoring

The last statement says that the product of the two factors is positive. Thus, either both factors are positive or both factors are negative (multiplication of two positive, or two negative factors, will result in a positive answer).

$$\begin{array}{ccc} (x+1) > 0 & \text{or} & (x+1) < 0 \\ \text{and} & & \text{and} \\ (x+3) > 0 & \text{or} & (x+3) < 0 \end{array}$$

Solving for x

$$x > -1 \quad \text{or} \quad x < -1$$
$$\text{and} \qquad\qquad \text{and}$$
$$x > -3 \quad \text{or} \quad x < -3$$

For all statements to be true for a given value of x, our solution would be:

$$x > -1 \quad \text{or} \quad x < -3$$

This states for all values of x greater than - 1, or less than - 3, our inequality is true.

Graphic Method

To solve the same inequality graphically, first transpose all terms to one side. Write the inequality as an equation. We introduce the variable y into the equation in place of zero (0), thus forming the equation of a parabola. Now we graph the parabola and determine our solution by inspection.

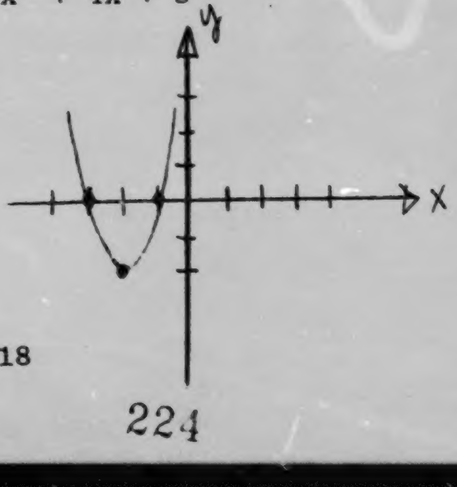
EXAMPLE: $x^2 + 3 > -4x$

$x^2 + 4x + 3 > 0$ transposition

Now we write the inequality as an equation of a parabola:

$$y = x^2 + 4x + 3$$

Graphing the parabola:



Remember $y = x^2 + 4x + 3$ and $x^2 + 4x + 3 > 0$.

So our solution would be the values of x when $y > 0$. Looking at the graph, $y > 0$ when $x < -3$ or when $x > -1$.

EXAMPLE: $x^2 < 4x - 4$

$x^2 - 4x + 4 < 0$ transposition

$(x - 2)^2 < 0$ factoring

The last statement is false for all Real values of x. Any Real value, squared, will never be less than zero. Therefore, the inequality has no solution.

If a quadratic expression in an inequality is difficult to factor, the graphical method is most convenient in order to determine existing solutions.

Solve and graph the following quadratic inequalities:

1) $x^2 + 5x \geq -6$

2) $x^2 < 16$

3) $x^2 \leq 2x + 8$

4) $0 \leq x^2 + 2x - 3$

5) $x^2 > -4x + 5$

FUNCTIONS

Basic Definition

One of the most important concepts in mathematics is the function. Sometimes, the word "mapping" is used to refer to a function.

To begin our explanation, the concept of "ordered pair" must be understood.

(a,b) is an ordered pair, where

a → first coordinate

b → second coordinate

We call (a, b) an "ordered pair" because it does make a difference which element comes first.

On a rectangular coordinate graph, with x and y axes, ordered pairs are listed (x, y). The x values, first coordinates, are the domain. And the y values, second coordinates, are listed in the range.

Using our knowledge of ordered pairs, a function is defined as:

Function → A set of ordered pairs in which each second coordinate has a unique first coordinate.

For example, consider the following set of ordered pairs:

$$F = \{(1, 2), (3, 5), (-1, 4), (0, 7)\}$$

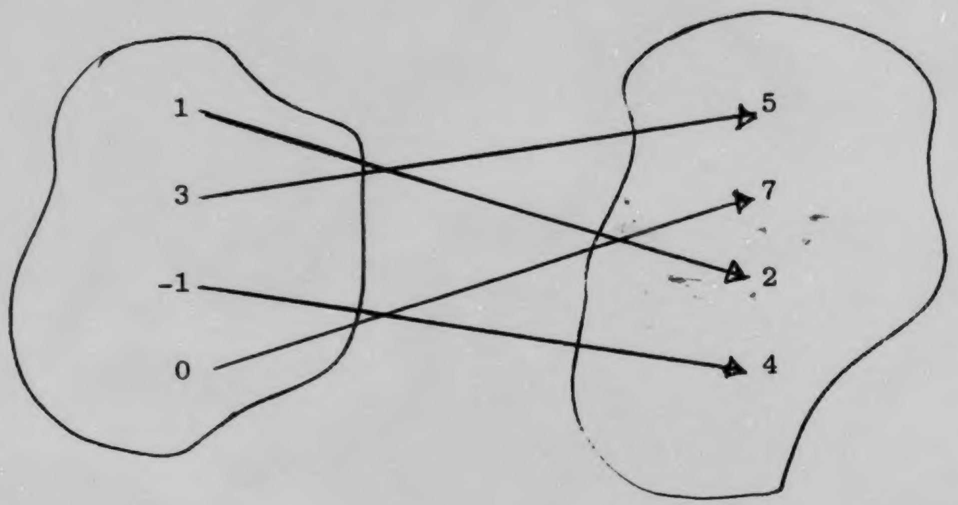
↑

We are naming this set, F.

Notice that each second coordinate of every ordered pair, has a unique first coordinate. Therefore, set F, is a function.

In the beginning definition, we stated that "functions" are sometimes called or associated with "mappings". To illustrate, use the previous mentioned set F:

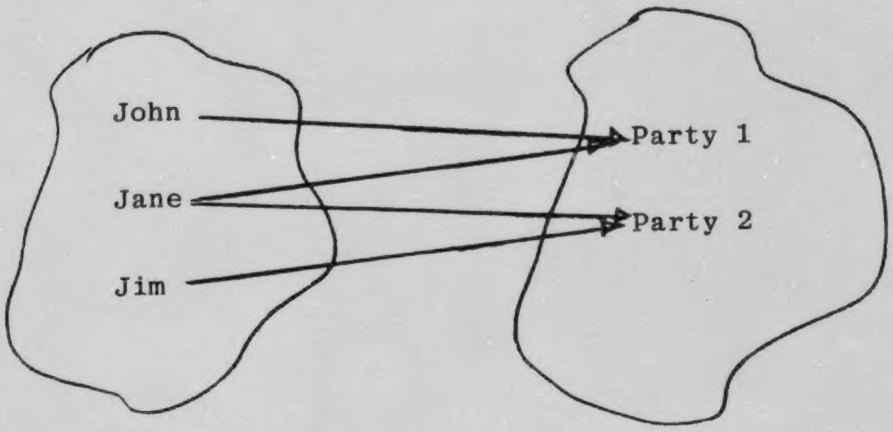
Mapping of Set F



In a function illustrated by a mapping, each element in the domain "maps" to an element in the range.

Suppose we have a set A composed of people attending parties at a particular time.

Mapping of Set A



In this mapping, John and Jane are going to party 1. Jim will attend party 2. However, Jane is also "mapped" to party 2 at the same time she is going to party 1.

This "mapping" does not indicate a function by definition. One person cannot attend two different parties at the same time. In the same sense, a first coordinate element cannot be mapped to two different second coordinate elements. If so, a function would not exist.

To further explain by set notation:

$$G = \{(1, 4), (5, -3), (1, 2)\}$$

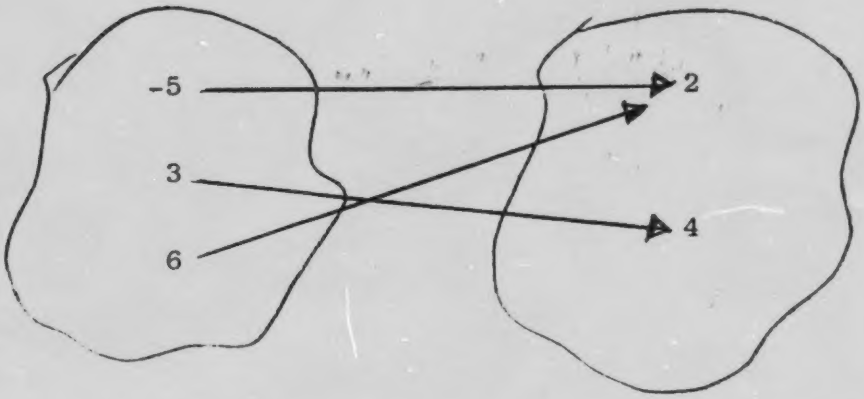
Set G is a set of three ordered pairs. However, the first coordinate 1 of (1, 4) is also mapped to 2, indicated by (1, 2). Set G is not a function

A function has just one second coordinate for each first coordinate.

We have just stated: "a function has just one second coordinate for each first coordinate." However, it is important to note at this time, a function can use a second coordinate many times. For example, given another set:

$$B = \{(-5, 2), (3, 4), (6, 2)\}$$

Set B is a function. By illustration:



The elements -5 and 6 are each mapped to the element 2.

As in the example of people attending parties, any number of people can go to the same party at one time. However, one person cannot go to two different parties at the same time.

To conclude, if all first coordinates in a set of ordered pairs are mapped to the same second coordinate, the function is constant.

For example:

$$C = \{(1, 2), (4, 2), (-6, 2), (\tfrac{1}{2}, 2)\}$$

Set C is a constant function.

Functional Notation

If we have an equation for a straight line, say $y = 2x + 4$, it is read "an equation of y in terms of x." This linear equation is a function. Each time a value of x is substituted, a value of y is obtained. For example:

$$y = 2x + 4$$

If $x = 0$; $y = 2(0) + 4 = 0 + 4 = 4$

$$y = 2x + 4$$

If $x = 1$; $y = 2(1) + 4 = 2 + 4 = 6$

$$y = 2x + 4$$

If $x = -1$; $y = 2(-1) + 4 = -2 + 4 = 2$

The ordered pairs obtained to this point are: (0, 4), (1, 6), and (-1, 2). Notice that each second coordinate has a unique first coordinate.

Because of this relationship within the equation, $y = 2x + 4$, y is referred to as a "function of x." The notation is:

$$y = f(x) = 2x + 4$$

Instead of calculations such as:

If $x = 0$; $y = 2(0) + 4 = 0 + 4 = 4$

We now do the following:

$$f(x) = 2x + 4$$

$$f(0) = 2(0) + 4 = 0 + 4 = 4 \rightarrow (0, 4)$$

$$f(1) = 2(1) + 4 = 2 + 4 = 6 \rightarrow (1, 6)$$

$$f(-1) = 2(-1) + 4 = -2 + 4 = 2 \rightarrow (-1, 2)$$

On a historical note, Leonhard Euler, born 1707 in Switzerland, made numerous contributions to mathematics. Euler is responsible for the symbol of functional notation, f(x).

Functions do not have to be named "f(x)". Choice of the letter "f" is optional. However, if the equation contains the variable x, then the letter x must be used. Refer to sample functions:

Equation	Functional Notation
$y = x - 3$	$f(x) = x - 3$
$y = 5X^2 + 2$	$g(x) = 5x^2 + 2$
$x = 2y - 3$	$h(y) = 2y - 3$
$A = 3c - 1$	$z(c) = 3c - 1$
$R = 2b + 3$	$f(b) = 2b + 3$
$C = x^2y + 3xy + 5$	$f(x,y) = x^2y + 3xy + 5$

Examples:

Let $k(x) = 3x^2 - 4$, then

$$k(0) = 3(0)^2 - 4 = 0 - 4 = -4$$

$$k(1) = 3(1)^2 - 4 = 3 - 4 = -1$$

$$\begin{aligned} k(4) &= 3(4)^2 - 4 = 3(16) - 4 \\ &= 48 - 4 \\ &= 44 \end{aligned}$$

Ordered pairs obtained to this point for $k(x)$:

$(0, -4)$, $(1, -1)$, $(4, 44)$

Let $p(y) = -2y + 3$, then

$$p(0) = -2(0) + 3 = 0 + 3 = 3$$

$$p(1) = -2(1) + 3 = -2 + 3 = 1$$

$$p(-5) = -2(-5) + 3 = 10 + 3 = 13$$

Ordered pairs obtained to this point for $p(y)$:

$(3, 0)$, $(1, 1)$, $(13, -5)$

Concluding our discussion on "what is a function", a simple definition would be:

> For every x assigned in a function, f(x), a unique result of f(x) is obtained.

Basic scientific formulas are functions. Consider:

Area of Square = s^2 $A = s^2$	Area of a square is a function of its side length, squared.
Area of Circle = $2\pi r$ $A = 2\pi r$	Area of a circle is a function of the constant π, multiplied by its radius and 2.
Voltage = (Current)(Resistance) $V = IR$	Voltage of a circuit is a function of amps and ohms (current and resistance)
$^\circ F = \frac{9^\circ C}{5} + 32$	Fahrenheit can be expressed as a function of $^\circ C$ multiplied by a constant of $\frac{9}{5}$ and 32 added.
Work Done = (Force)(Distance)	Work to accomplish a movement is a function of force times distance.

Determine if each given set of ordered pairs is a function by definition:

1) B = { (1, 2), (3, 5), (2, 7) }

2) A = { (-2, 1), (3, -5), (4, 6) }

3) Z = { (5, 10), (2, -1), (3, 4), (2, 8) }

4) Q = { (1, 3), (5, 3), (-6, 3), (4, 3) }

Also, for each function in the above list, state the domain and range.

Evaluate each function for integer values of x from $x = -2$ through $x = 3$.

1) $f(x) = x - 3$

2) $g(x) = 5x^2 + 2$

3) $h(x) = 2x - 3$

4) $z(x) = 3x - 1$

5) $s(x) = 2x + 3$

Evaluate each function for integer values of y from $y = -1$ through $y = 1$.

1) $h(y) = 2y^2 - 3$

2) $f(y) = 3y + 1$

3) $k(y) = -4y + 6y^2$

4) $g(y) = y + 3$

5) $z(y) = -2y - 5$

FUNCTIONS

1) Given the following information,

$$g(x) = x^2 + 2x + 1 \qquad f(x) = 3x - 5$$

Find:

a. $f(0)$

b. $f(5)$

c. $g(-2)$

d. $g(3)$

e. $f(a)$

f. $g(a^2)$

g. $f(x + h)$

h. $g(x + h)$

2) Given the following information,

$$f(x) = 3x - 1$$

Find the missing value for the following points.

a. $(?, 2)$

b. $(?, 14)$

c. $(7, ?)$

d. $(-2, ?)$

e. $(?, -10)$

f. $(10, ?)$

g. $(0, ?)$

h. $(?, a)$

Find the horizontal and vertical components for the following vectors:

1) $\underline{F} = 5$ lbs. $\theta = 30^\circ$

2) $\underline{F} = 7$ lbs. $\theta = 60^\circ$

3) $\underline{F} = 6$ lbs. $\theta = 45^\circ$

4) $\underline{F} = 15$ lbs. $\theta = 135^\circ$

5) $\underline{F} = 7$ lbs. $\theta = 0^\circ$

1) Suppose a force of 250 kg is acting at an angle 31° 30' with the horizontal. Find the horizontal and vertical components.

2) Suppose a force of 100 g is acting at an angle 29° 15' with the vertical. Find the horizontal and vertical components.

3) A block weighing 25 kg is at rest on an incline of 45° with the horizontal. What is the force of the block on the incline?

GEOMETRY

Chapter III - Geometry

1. An Introduction
2. Area
3. $\pi = 3.1415...$
4. Volume
5. Conic Sections
6. Conic Sections - Introduction to Applications

GEOMETRY

An Introduction

The ancient Greeks believed the study of Geometry most important. Points and line segments represented numbers and variables.

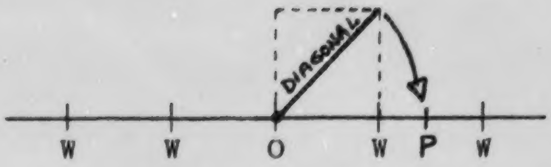
It was this type of geometric interpretation which led to interesting discoveries by the Greeks, such as the Irrational Numbers.

Remembering from the Real Number System, rationsl numbers are defined as the quotient of two integers:

$$\frac{a}{b}, \text{ where } b \neq 0$$

The geometric interpretation of rational numbers is very simple -- for each rational number, there exists a point on a number line.

The following diagram explains through the use of Geometry, how the irrational numbers were discovered.



O → Origin

W → Whole Number

P → Point Not Represented by a Whole Number

A square has been constructed on a number line, with points W representing Whole Numbers. If the diagonal of the square is rotated to lay on the number line, the end of the diagonal creates a new point P, between two whole numbers.

Therefore, numbers had to be invented to correspond with points such as these -- the irrationals.

The ancient Greeks noticed that mathematics occurred in practically every field of study, from art and music to philosophy. Because of this, it was thought to have a divine purpose and considered the linkage between religion and science.

The Academy in Greece, founded by Plato, had the following inscription above its entrance:

"LET NO MAN UNVERSED IN GEOMETRY ENTER HERE."

However, Plato was impressed with the intelligence of any individual, man or woman. Women attended speeches at The Academy, even though groups of women in public places was against the law. These enlightening lectures inspired several women to further study mathematics, leading to highly respected manuscript contributions.

Discussion of Geometry would be incomplete if we did not mention Euclid. A professor of mathematics at the University of Alexandria (about 300 B.C.), and former student of the Pythagorean school, Euclid is credited with at least ten works, the Elements being the most popular.

The Elements consisted of several manuscripts, or books. Book I contains geometric definitions that we use today. Books II and III discuss area and law of cosines, circles and angles. Book IV deals with the construction of polygons -- closed plane (two-dimensional) figures bounded by three or more line segments. Book V presents an exposé on the theory of proportion, first introduced by Eudoxes, a pupil of Plato. The theory states, "a line parallel to one side of a triangle divides the other two sides proportionally." Remaining books of the Elements deal with applications of proportion, number theory, and some elementary algebra.

Eudoxes' theory included irrational numbers, other than $\sqrt{2}$, thought at the time to be the only irrational number. The philosophy of the Pythagorean school in ancient Greece was based upon mysticism and the premise of whole numbers controlling not only the universe, but man's destiny. Though the discovery of $\sqrt{2}$ was considered a great achievement, introductions of other irrationals caused mixed, suspicious reaction.

Among the major Greek contributors to Geometry, was Hypatia (370 - 415), daughter of a University of Alexandria professor. Hypatia was also a teacher at the university and well respected by her students. Several works by Hypatia include a critique on Euclid, and a review of the conic sections, first introduced by Apollonius.

The conic sections are utilized most frequently in modern day mathematics. It is interesting to note that further study of the hyperbola, parabola, and ellipse did not occur until the 17th century.

AREA

The concept of area, measurement of a plane (two-dimensional) region, is a simple procedure. However, the calculation of area in ancient times, especially in reference to the circle, was the source of many disagreements.

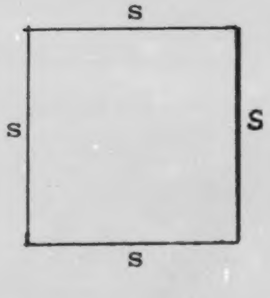
When measuring a two-dimensional region enclosed by unit lengths, the result is "square units" of region. For instance, the formula to determine the area of a rectangle is, A = (length)(width). Because length and width each has measurement, the multiplication of the two amounts will produce either square inches (in^2), square feet (ft^2), square meters (m^2), etc. Example:

$$\text{Area of Rectangle} = (4 \text{ inches})(6 \text{ inches})$$
$$= 24 \text{ square inches}$$
$$= 24 \text{ in}^2$$

The following represent the most common figures and their corresponding formulas to calculate area:

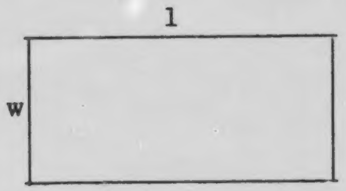
Square

Area = s^2, where



s → length of one side

Rectangle

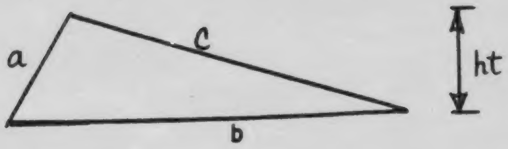


Area = (l)(w)

l → length

w → width

Triangle



Area = $\frac{1}{2}$ (base)(ht)

a → length of one side

b → base length of triangle

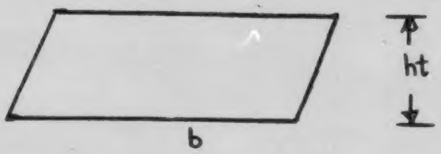
c → length of one side

h → height, also known as altitude

Note: If the height, or altitude of a triangle is not known, but the three sides, a, b, c are given, then:

$$\text{Area} = \frac{1}{2}(a + b + c)$$

Parallelogram

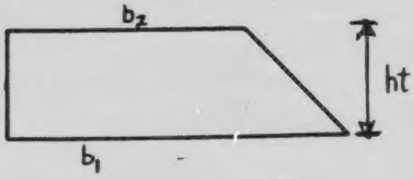


Area = (base)(height)

h → height, or altitude

b → base

Trapezoid



Area = $(h)(b_1 + b_2)$

h → height or altitude

b_1 → lower base

b_2 → upper base

Find the area given the following information:

1) Rectangle $l = 5$ $w = 6$

2) Square $l = 6$

3) Parallelogram $b = 7$ $h = 6$

4) Trapezoid $h = 6$ $b_1 = 3$ $b_2 = 9$

5) Circle $r = 3$

$\pi \rightarrow$ PI = 3.1415....

Every time a circular measurement occurs, the constant of π is involved. The approximate value of π was known in ancient times in the form of a ratio. Why does the value π enter any calculations of circular measure?

An interesting concept to see the occurrence of π is by comparing circles of different size. For instance, suppose we had four circles of varying radius, and Circumference, C:

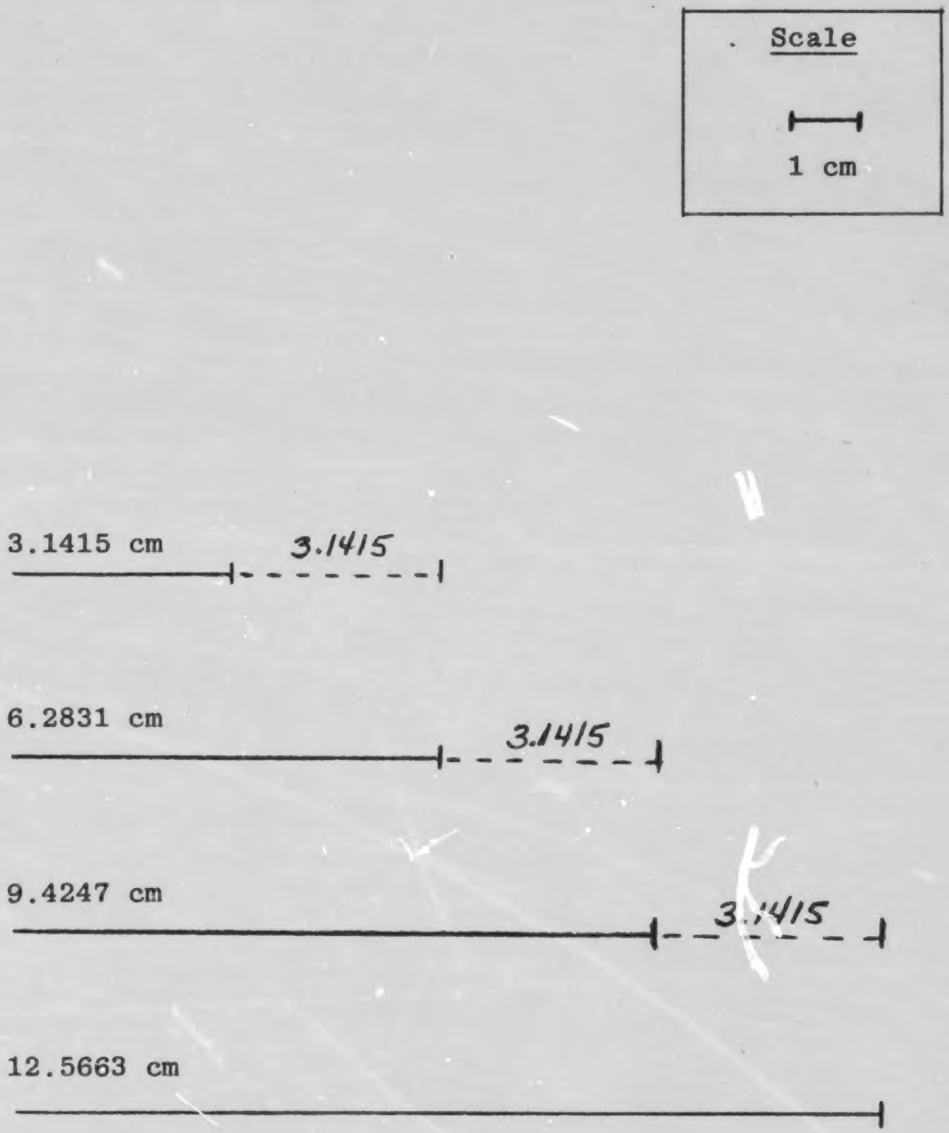
Circle I. $r = 0.5$ in, then
$C = 2\pi r = 2\,(3.1415)(0.5\text{ cm}) = 3.1415\text{ cm}$

Circle II: $r = 1.0$ in, then
$C = 2\pi r = 2\,(3.1415)(1.0\text{ cm}) = 6.2831\text{ cm}$

Circle III: $r = 1.5$ in, then
$C = 1\pi r = 2\,(3.1415)(1.5\text{ cm}) = 9.4247\text{ cm}$

Circle IV: $r = 2{,}0$ in, then
$C = 2\pi r = 2\,(3.1415)(2.0\text{ cm}) = 12.5663\text{ cm}$

Also, note the illustration below. Pretend each circle has been stretched into a straight line:


Scale
1 cm
3.1415 cm
3.1415
6.2831 cm
3.1415
9.4247 cm
3.1415
12.5663 cm

From the end point of one circle to the endpoint of the next circle, a factor of approximately 3.1415 cm occurs. The unit of (cm) is in this case only. Specifically, 3.1415 occurs each time, whether the units are in feet, inches, yards, etc.

Therefore, the value of π, 3.1415..., is actually the ratio between the circumference of a circle, to its diameter.

Compare other circles having varying radius.

VOLUME

Volume is space occupied in 3 - dimensions and is measured in cubic units. Common units of volume are:

Cubic Inch = in^3

Cubic Centimeter = cm^3 also known as cc

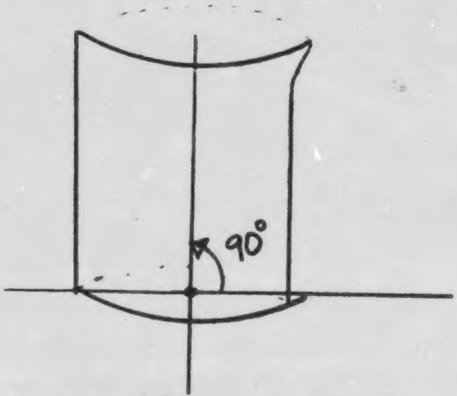
Cubic Meter = m^3

Wet measurements, in small quantities, are made in calibrated glass cylinders. Calibrated means that the measuring instrument being used has the units specified in equal divisions.

For example, a 125 ml flask, commonly used in chemistry labs, is calibrated into 25 ml. The flask is marked in equal divisions of 25 ml; 50, 75, 100, and then 125 ml.

Measuring instruments used most of the time are RIGHT CIRCULAR CYLINDERS.

Example:



It is called this because the point at the center of the base is directly under the center of the top. Imaginary lines have been drawn through the base center point. The intersection of the lines form a right, or 90°, angle.

Remember, the definition of volume: "Space occupied in 3 - dimensions." In the case of a right circular cylinder, space is within the bounds of its height, diameter, and circumference (distance around the cylinder).

To determine the volume of any right circular cylinder, we use the formula:

$$\text{Volume} = (\pi)(r^2)(h)$$

Where $\pi \doteq 3.1415$

r = radius of the cylinder

h = height of the cylinder

PROBLEM:

If a right circular cylinder has dimensions, radius = 6 inches and height = 21 inches, determine its capacity of volume:

SOLUTION:

$$\text{Volume} = (\pi)(r^2)(h)$$

$$V = (3.1415)(6\text{ in})^2(21\text{ in})$$

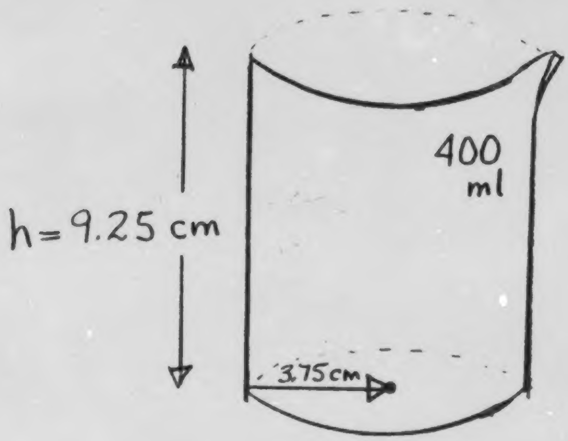
$$V = (3.1415)(36\text{ in}^2)(21\text{ in})$$

$$V = 2{,}374.97\text{ in}^3$$

This means a right circular cylinder having these particular dimensions could hold 2,374.97 in^3 of liquid.

An interesting point to mention is "how did the manufacturer know what dimensions would produce a glass cylinder to hold 400 ml?"

This process involves a little more advanced math than would be proper to introduce here. But, we can measure the dimensions of a manufactured 400 ml glass container, and see if the volume, by formula, is in fact 400 ml.



Using an ordinary ruler, the height of the 400 ml glass cylinder is about $9\frac{1}{4}$ cm, or 9.25 cm. The radius of the cylinder, which is half the diameter, is about $3\frac{3}{4}$ cm, 3.75 cm. Now, use the formula for volume: Volume = $(\pi)(r^2)(h)$

$$V = (3.1415)(3.75\text{ cm})^2(9.25\text{ cm})$$

$$V = (3.1415)(14.0625\text{ cm}^2)(9.25\text{ cm})$$

$$V \doteq 408.64\text{ cm}^3$$

Remember, since we measured the cylinder with a ruler, and obtained approximate measurements, the final Volume would also be approximate. Meaning, instead of obtaining exactly 400 ml, we calculated an approximate volume, 408.64 cm^3.

Also, we are looking for units of ml, and not cm^3. Let's see what cm^3 actually means.

Using the conversion factor,

$$1000 \text{ cm}^3 = 1 \text{ liter}$$

To find how many cm^3 are in 1 ml, divide both sides of the conversion factor by 1000. We do this because the prefix, milli- implies one-thousandth of a unit:

$$\frac{1000 \text{ cm}^3}{1000} = \frac{1 \ \ell}{1000}$$

Therefore,

$$\frac{1000 \text{ cm}^3}{1000} = \frac{1 \ \ell}{1000}$$

$$1 \text{ cm}^3 = \frac{1 \ \ell}{1000} = 1 \text{ ml}$$

$$1 \text{ cm}^3 = 1 \text{ ml}$$

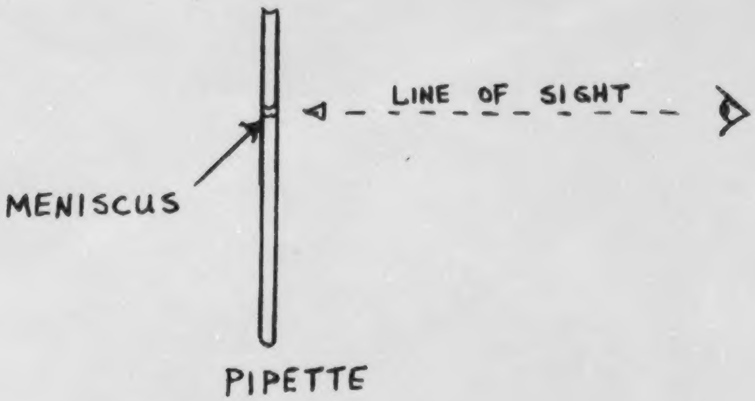
It has been shown:

$$\boxed{1 \text{ cm}^3 = 1 \text{ ml}}$$

NOTE: The importance of accurately measuring decimals and fractions cannot be understated. Our problem was only to show that a particular right circular cylinder would hold about

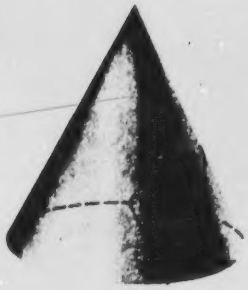
400 ml; the answer was an approximation. However, the slightest decimal or fractional error, would creat a possible disaster in a chemistry lab.

Instruments used to measure very small quantities of liquids, are called pipettes. In containers such as these, having a small cross section, other factors come into play while measuring volume.



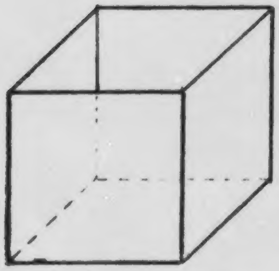
The meniscus is formed as liquid in the cylinder is drawn up at the edges, with the liquid touching the inside of glass. Line of sight of the observer must be perpendicular to the pipette. If not, there is an error in reading the volume, which is referred to as a parallax.

We have limited our discussion to explaining what volume is and how it is applied to right circular cylinders. But this is not the only cylinder having volume. Note the following:



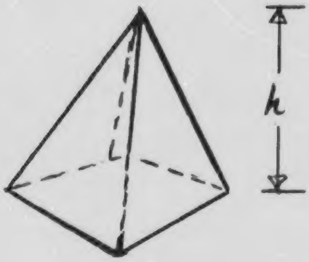
$$\text{Volume}_{\text{cone}} = \frac{(\pi)(r^2)(h)}{3}$$

CONE



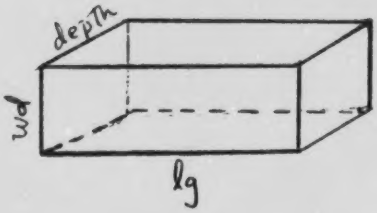
$$\text{Volume}_{\text{cube}} = (\text{length of an edge})^3$$

CUBE



$$\text{Volume}_{\text{pyramid}} = \frac{(\text{Area of Base})(h)}{3}$$

PYRAMID



$$\text{Volume}_{\text{rectangle}} = (\text{lg})(\text{wd})(\text{depth})$$

RECTANGLE

$$\text{Volume}_{\text{sphere}} = \frac{4(\pi)(r^3)}{3}$$

SPHERE

Find the volume given the following information:

1) cylinder r = 3 h = 8

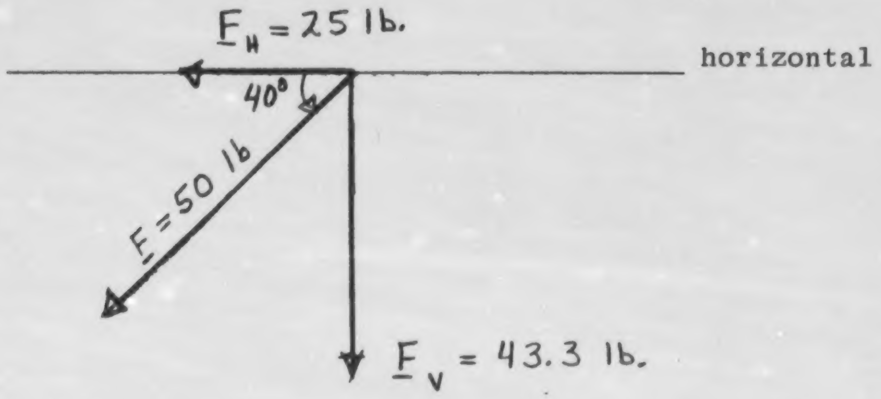
2) cone r = 6 h = 9

3) cube l = 2 w = 3 b = 7

4) pyramid l = 3 w = 6 h = 4

5) sphere r = 3

By illustration, the components are:



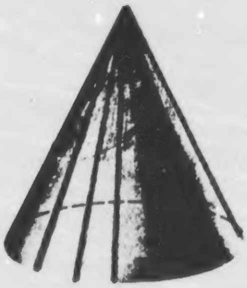
$\underline{F}_H = 25$ lb.
horizontal
40°
$\underline{F} = 50$ lb
$\underline{F}_V = 43.3$ lb.

CONIC SECTIONS

An Introduction

In this chapter, we will investigate some curves graphed from second-degree, or quadratic, equations in two variables. These curves are called the conic sections. Why? Because the curves are derived from a right circular cone.

Here is a right circular cone:

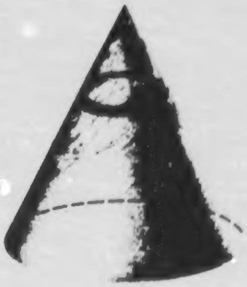


The straight lines are referred to as generators. Actually, they are imaginary lines, revolving about a vertex, forming a cone. The concept of generators is important. To form the curves known as conic sections, a plane is passed through the cone at an angle dependent upon the generators.

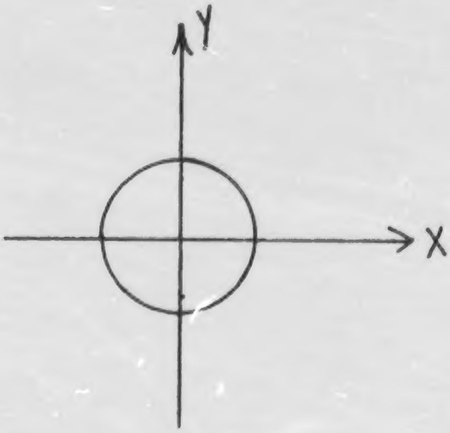
When we speak of a "plane", we mean a 2-dimensional figure. A figure having length and width. Our examples will utilize a sheet of paper to represent a "plane". Though the edge of the paper could be measured---it would be very small, perhaps micro-units. So for all practical purposes of the illustration, we consider the paper having length and width only.

Circles

If we could pass this sheet of paper thru the cone, parallel with the base, we would cut into the cone, forming a circle. As long as the plane is parallel to the base, we could cut into the cone at any height, and still obtain a circle.



If we wanted to graph, on paper, this circle, we use the x and y axes.



A circle is a conic section, which is stated as a second-degree, quadratic equation of the form:

$$Ax^2 + By^2 = C, \text{ where } A^2 + B^2 \neq 0$$

And if A = B, with A, B, and C having like signs.

It's easy to sketch curves from an equation of this type.

EXAMPLE:

$$\{(x,y) \mid x^2 + y^2 = 4\}$$

This is a quadratic equation written in set notation. The brackets, {}, just package the equation. The (x,y) denotes there are two variables, named x and y, in this particular equation. The vertical line | denotes "such that". And then, we have the equation.

If we were asked to sketch the graph of $x^2 + y^2 = 4$, first compare the given equation with the standard form of a circle:

$$Ax^2 + By^2 = C \qquad A^2 + B^2 \neq 0$$

$$x^2 + y^2 = 4$$

Signs of the coefficients of x^2 and y^2, 1 and 1, respectively, and the sign of C, are alike. If we square each coefficient and then add the two results, we get 2. This satisfies the other condition required in the circle equation, $A^2 + B^2 \neq 0$.

If we let $x = 0$, and solve for y, we have:

$$\begin{aligned} x^2 + y^2 &= 4 \\ 0 + y^2 &= 4 \\ y^2 &= 4 \\ y &= \pm\sqrt{4} \\ y &= +2 \text{ and } y = -2 \end{aligned}$$

$$\underset{(0,\ 2)}{\wedge} \qquad\qquad \underset{(0,\ -2)}{\wedge}$$

If y = 0 and we solve for x, then:

$$x^2 + y^2 = 4$$

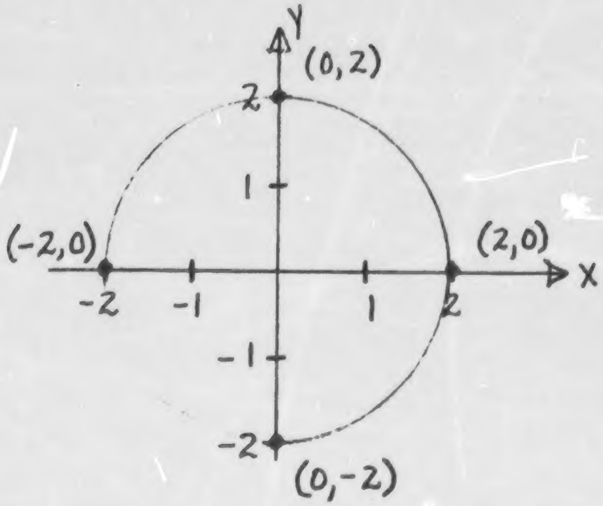
$$x^2 + 0 = 4$$

$$x^2 = 4$$

$$x = \pm\sqrt{4}$$

$$x \underset{\overset{\wedge}{(2,\ 0)}}{=} +2 \text{ and } x = \underset{\overset{\wedge}{(-2,\ 0)}}{-2}$$

Now, we can sketch our figure because we have obtained four points:



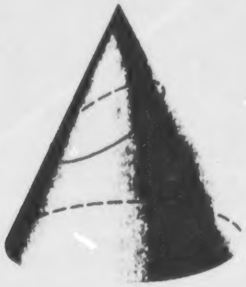
Sometimes, the equation of a circle is expressed:

$$Ax^2 + By^2 = r^2 \quad , A^2 + B^2 \neq 0$$

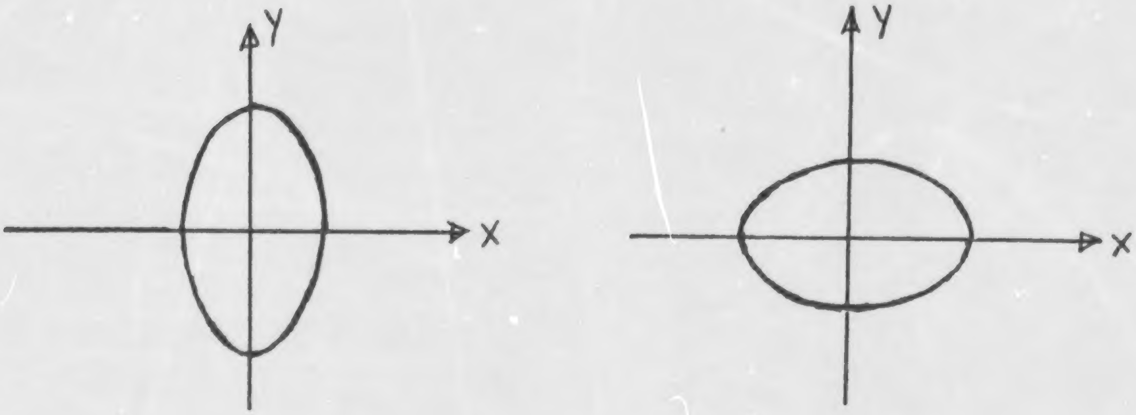
Where r → radius.

Ellipse

Referring back to the right circular cone, suppose we passed our sheet of paper, the plane, through the cone at an angle. Not just any angle. Remember the generators in the cone? In this case, we will pass the sheet through the cone at an angle which is not parallel to a generator. Doing this will produce a curve known as an ellipse.



Roughly sketching an ellipse on paper:



The figure gives an impression of a slightly, squashed circle.

There are many applications of the ellipse occurring in a variety of fields. In astronomy, planets, satellites, and moons orbit in an elliptical fashion. Machine gears and parts utilize the shape, as well as construction of overpasses and bridges.

If we have a quadratic equation of the form:

$$Ax^2 + By^2 = C \quad , \text{ where } A^2 + B^2 \neq 0$$

The equation will sketch an ellipse, if and only if:

$A \neq B$, and A, B, and C have like signs.

EXAMPLE:

$$\{(x_1 y) \mid 3x^2 + 25y^2 = 100\}$$

To sketch the curve associated with this equation, first compare it to the equation form and conditions required for an ellipse.

$$Ax + By = C, \qquad A + B \neq 0, \quad A \neq B$$

$$3x + 25y = 100 \qquad \text{A, B, and C have like signs}$$

The coefficients of x^2 and y^2, are 3 and 25 respectively. Therefore, they are unequal to each other, A, B, and C in the problem; 3, 25, and 10, have like signs.

The quadratic equation satisfies all the conditions required for an ellipse.

If we let x = 0, and solve for y:

$$\begin{aligned}
3x^2 + 25y^2 &= 100 \\
3(0)^2 + 25y^2 &= 100 \\
25y^2 &= 100 \\
y^2 &= 4 \\
y &= \pm 4 \\
y &= +2 \text{ and } y = -2
\end{aligned}$$

Next, if we let y = 0, and then solve for x:

$$\begin{aligned}
3x^2 + 25y^2 &= 100 \\
3x^2 + 25(0)^2 &= 100 \\
3x^2 &= 100 \\
x^2 &= \frac{100}{3} \\
x &= \pm\sqrt{\frac{100}{3}} \\
x &= \pm \frac{10}{\sqrt{3}} \\
x &\doteq \pm \frac{10}{1.73} \\
x &\doteq \pm 5.6 \\
x &\doteq + 5.6
\end{aligned}$$

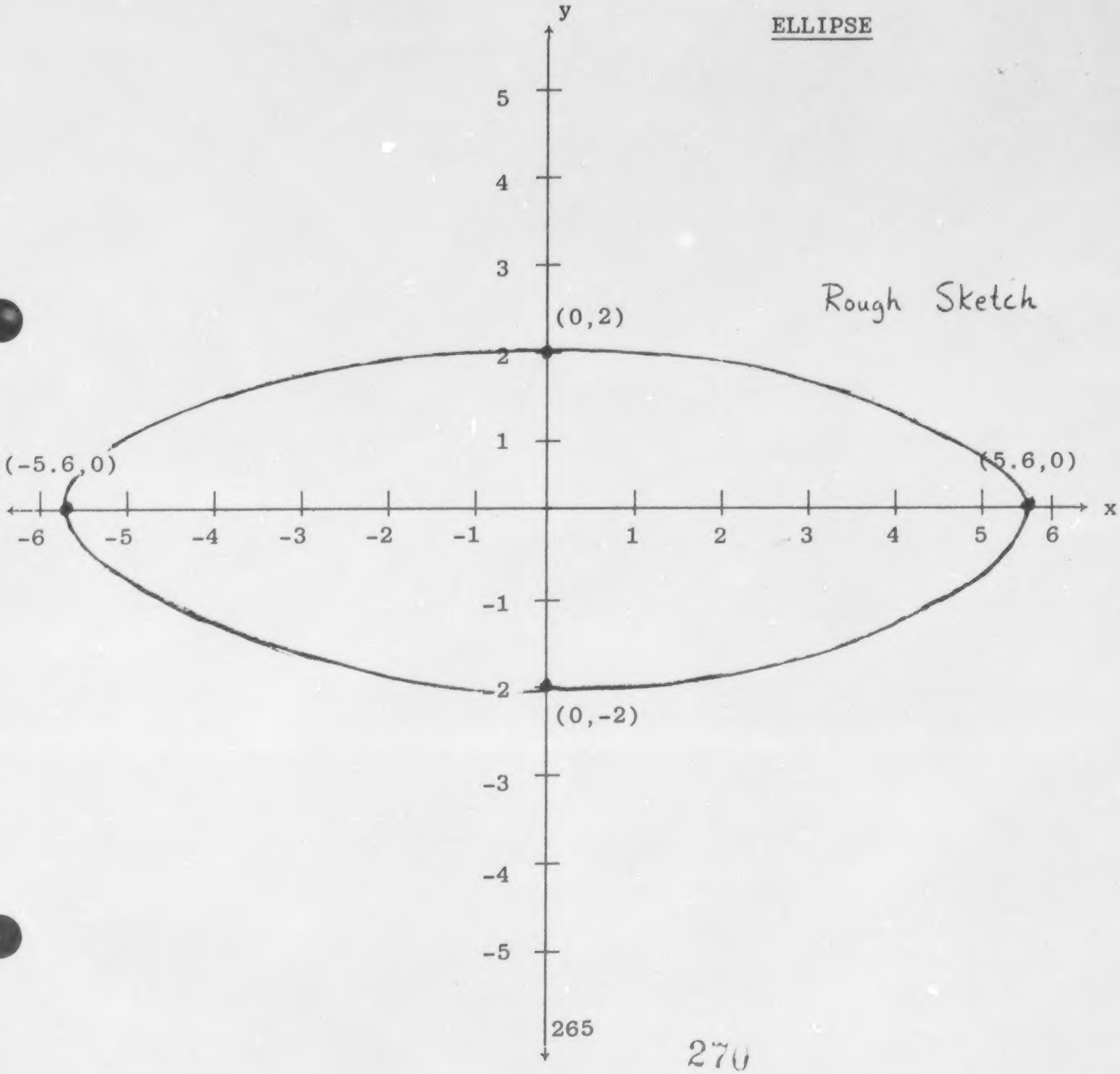
and

$$x \doteq - 5.6$$

We have found the solutions:

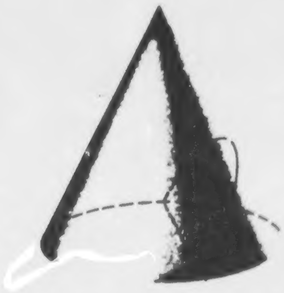
(0, 2), (0, -2), (5.6, 0), (-5.6, 0)

Sketching from these pairs:



Parabola

If the plane cuts through the cone parallel to one and only one generator of the cone, it is called a parabola. As with the ellipse, the shape of parabolas are often found in nature. Paths of projectiles, the construction of mirrors, determining 'how curved' an automobile headlight should be in relation to the car design, etc.

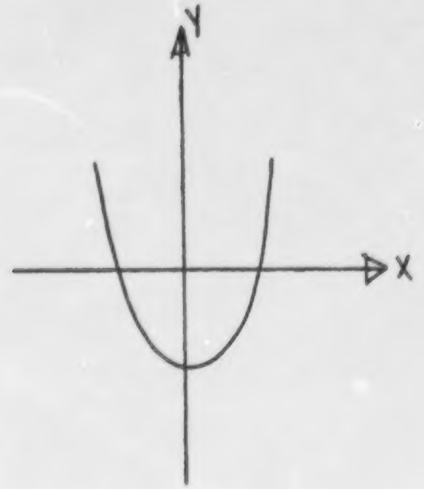


To recognize when a quadratic equation will graph a parabola, first notice the form:

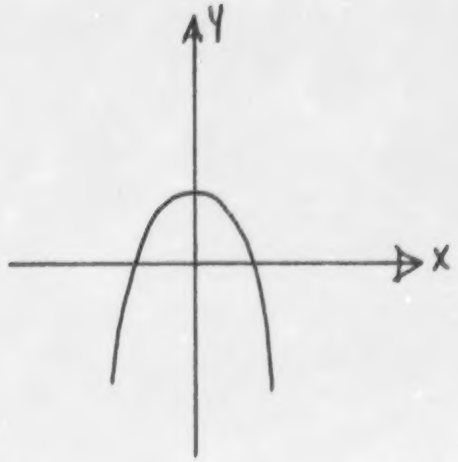
$$y = ax^2 + bx + c, \text{ where } a \neq 0$$

If a, the coefficient of x^2, is greater than zero, the parabola will open upward. Likewise, if a is less than zero, the parabola will open downward.

To further explain by diagrams:



Parabola: $y = ax^2 + bx + c$,
where $a \neq 0$,
and $a > 0$.



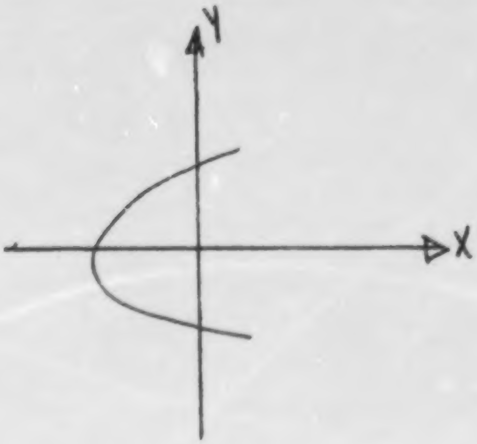
Parabola: $y = ax^2 + bx + c$,
where $a \neq 0$,
and $a < 0$.

The vertex, tip of the parabola is on the y axis.

If we have an equation of the form:

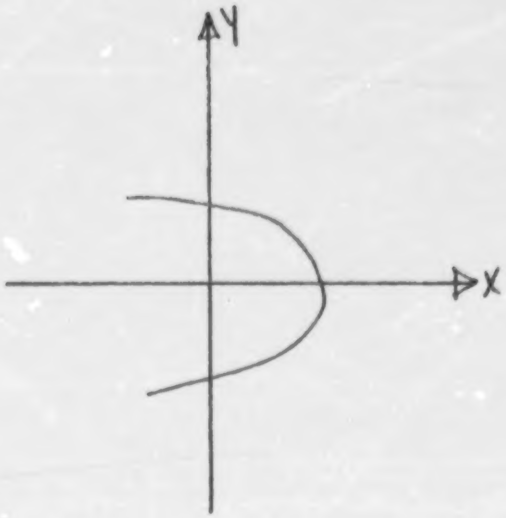
$$x = ay^2 + by + c, \quad a \neq 0$$

the parabolas sketched will look like the following:



Parabola: $x = ay^2 + by + c$,

$a \neq 0, \quad a > 0$



Parabola: $x = ay^2 + by + c$,

$a \neq 0, \quad a < 0$

In this case, the vertex, tip of the parabola, is on the x axis.

An example,

$$\{(x,y) \mid y = x^2 - 4\}$$

How can we determine this quadratic equation will sketch a parabola? According to our definition, the coefficient of x^2 must be unequal to zero. In this problem, the coefficient of x^2 is 1, which $1 \neq 0$.

One method of graphing a parabola, is to substitute arbitrary values for x, and solve for y.

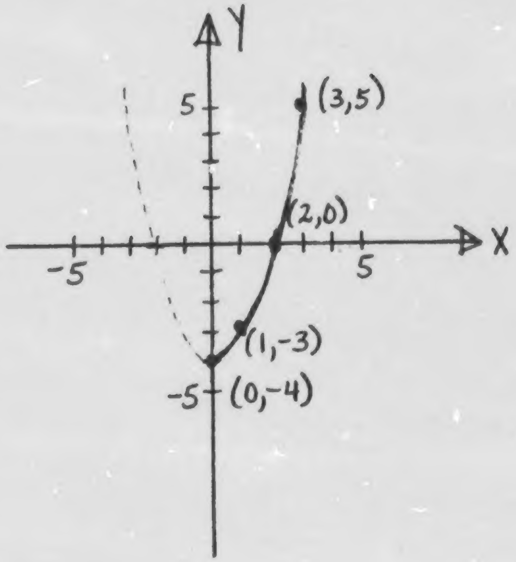
$$\text{If } x = 0;\quad \left.\begin{array}{l} y = x^2 - 4 \\ y = 0 \ - 4 \\ y = \ \ - 4 \end{array}\right\} \qquad (0, -4)$$

$$\text{If } x = 1;\quad \left.\begin{array}{l} y = x^2 - 4 \\ y = (1)^2 - 4 \\ y = 1 - 4 \\ y = - 3 \end{array}\right\} \qquad (1, -3)$$

$$\text{If } x = 2;\quad \left.\begin{array}{l} y = x^2 - 4 \\ y = (2)^2 - 4 \\ Y = 4 - 4 \\ y = 0 \end{array}\right\} \qquad (2, 0)$$

$$\text{If } x = 3;\quad \left.\begin{array}{l} y = x^2 - 4 \\ y = (3)^2 - 4 \\ y = 9 - 4 \\ y = 5 \end{array}\right\} \qquad (3, 5)$$

With these points, let's see if a parabola will be formed.



The points form a curve which resembles the right side of a parabola. The other side will be the gight's mirror image; a dotted curve was drawn to represent the image. To actually check the image with points, let x equal the negative counterparts of your previous x values; i.e., x = -1, s = -2, x = -3. Now, determine the solutions and check.

Another method of sketching a parabola involves solving only for the vertex. And, noticing from the coeffecient of the x^2 term if the parabola is concave upward or down.

For example:

$$\{(x, y) \mid y = x^2 - 4\}$$

In standard, quadratic form, the equation should be:

$$y = x^2 + 0x - 4$$

The 0x implies there is no middle term.

Location of Vertex

Since the equation is of the type, y = x terms + constant, the vertex is located on the y axis.

The x coordinate of the vertex is found by:

$$\text{Vertex}_x = \frac{-b}{2a}$$

In our problem, $y = x^2 + 0x - 4$, $a = 1$ and $b = 0$.

Therefore

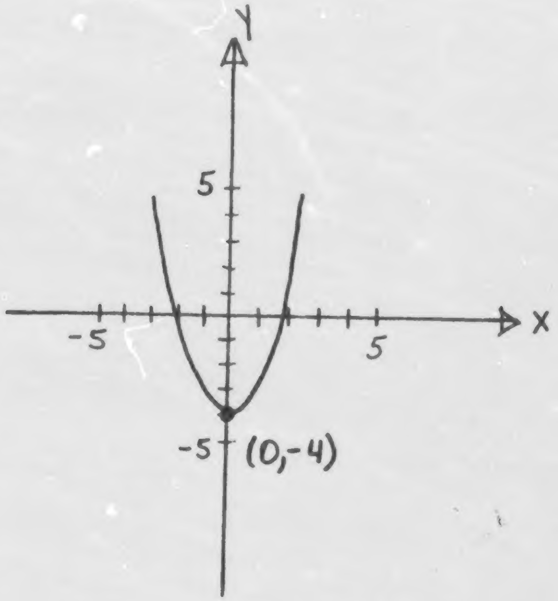
$$V_x = \frac{-0}{2(1)} = 0$$

To solve for the y coordinate of the vertex, substitute V_x into the original equation:

$$V_y \rightarrow y = x^2 - 4$$

$$y = (0)^2 - 4 = -4$$

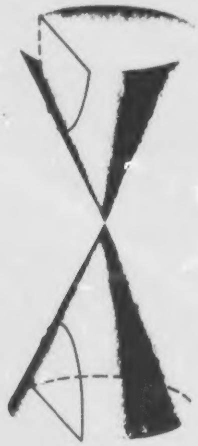
Therefore, the point of vertex is at (0, -4). Since the coefficient of x^2 is 1, implying greater than zero, the parabola will be concave upward. Note:



The graph is not as accurate as plotting points, but it gives an idea of how the parabola will look.

Hyperbola

To accurately define the hyperbola, two right circular cylinders would be necessary. For instance,



If our plane is cut through the cones parallel to *two* generators, we have two curves which form the *hyperbola*. This curve is found in mathematical applications. An example would be the graph used to interpret Boyle's law for a perfect gas; the hyperbola would represent pressure and volume. Hyperbolas are also used to locate objects by the sounds being made by the objects.

The second-degree (quadratic) equation yielding a curve known as the hyperbola, with its center at the origin is:

$$\{x,y) \mid Ax^2 - By^2 = C\}$$

where $A > 0$
> 0
> 0

Also, the equation can also be of the form:

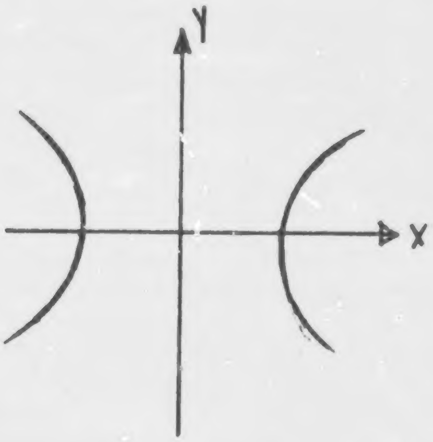
$$\{(x,y) \mid By^2 - Ax^2 = C\}$$

where $A > 0$

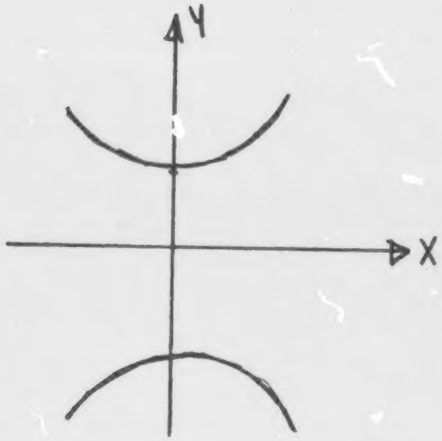
$B > 0$

$C > 0$

Generally, hyperbolas will sketch as:



or



EXAMPLE:

Suppose we have a quadratic equation:

$$\{(x,y) \mid x^2 - y^2 = 9\}$$

To see if this equation will graph as a hyperbola, compare with the defining equation:

$$\{(x,y) \mid Ax^2 - By^2 = C\} \quad \text{where } A, B, C > 0$$

$$\{(x,y) \mid x^2 - y^2 = 9\} \quad \text{where } A = 1$$

$$B = -1$$

$$C = 9$$

The coefficients of x^2 and y^2, are greater than zero, as well as the constant. Also, the negative sign is present between the two terms, x^2 and y^2.

Since our comparison is true, we are now ready to graph the hyperbola.

The easiest way is to first let $x = 0$ and then solve for y. Next, let $y = 0$ and solve for x.

Let $x = 0$:

$$\begin{aligned} x^2 - y^2 &= 9 \\ (0)^2 - y^2 &= 9 \\ - y^2 &= 9 \\ y^2 &= -9 \\ y^2 &= -9 \end{aligned}$$

STOP! The square root of a negative number cannot be taken in the field of Real numbers. Therefore, in this graph, the hyperbola cannot cross the y axis.

Let $y = 0$:

$$\begin{aligned} x^2 - y^2 &= 9 \\ x^2 - (0)^2 &= 9 \\ x^2 &= 9 \\ x^2 &= \pm\sqrt{9} \\ x &= \pm 3 \end{aligned}$$

Points are:
(3,0) and (-3,0)

Since we need another set of points, and possibly more, to aid in graphing the hyperbola, choose some arbitrary values for x and then solve for y. Remember, it was found that when $x = 0$, y did not have a real number value. So, let's choose:

Let $x = 4$:

$$\begin{aligned} x^2 - y^2 &= 9 \\ (4)^2 - y^2 &= 9 \\ 16 - y^2 &= 9 \\ -y^2 &= -7 \\ y^2 &= 7 \\ y^2 &= 7 \\ y &= \pm\sqrt{7} \end{aligned}$$

Points:
$(4, \sqrt{7})$
$(4, -\sqrt{7})$

Let $x = -4$:

$$\begin{aligned} x^2 - y^2 &= 9 \\ (-4)^2 - y^2 &= 9 \\ 16 - y^2 &= 9 \\ -y^2 &= -7 \\ y^2 &= 7 \\ y &= \pm\sqrt{7} \end{aligned}$$

Points:
$(-4, \sqrt{7})$
$(-4, -\sqrt{7})$

Therefore, we have found the following points:

$(3, 0), (-3, 0), (4, \sqrt{7}), (4, -\sqrt{7}), (-4, \sqrt{7}), (-4, -\sqrt{7})$

And, since the hyperbola cannot cross the y - axis, dotted lines have been drawn through the origin of the x, y axes.

To further explain the "dotted" lines, by definition in Analytic Geometry they are known as ASYMPTOTES. The hyperbolic curve approaches the asymptotes continually, but never touch.

A special type of conic section is the equilateral hyperbola which has an equation of the form:

$$xy = \text{a non zero constant}$$

Notice, the equation is not of second-degree (quadratic). However, a hyperbola will result on a graph. This is why it is referred to as a special type.

For example, let's consider:

$$xy = 18$$

Solving for x in terms of y, and then solving for y in terms of x would result:

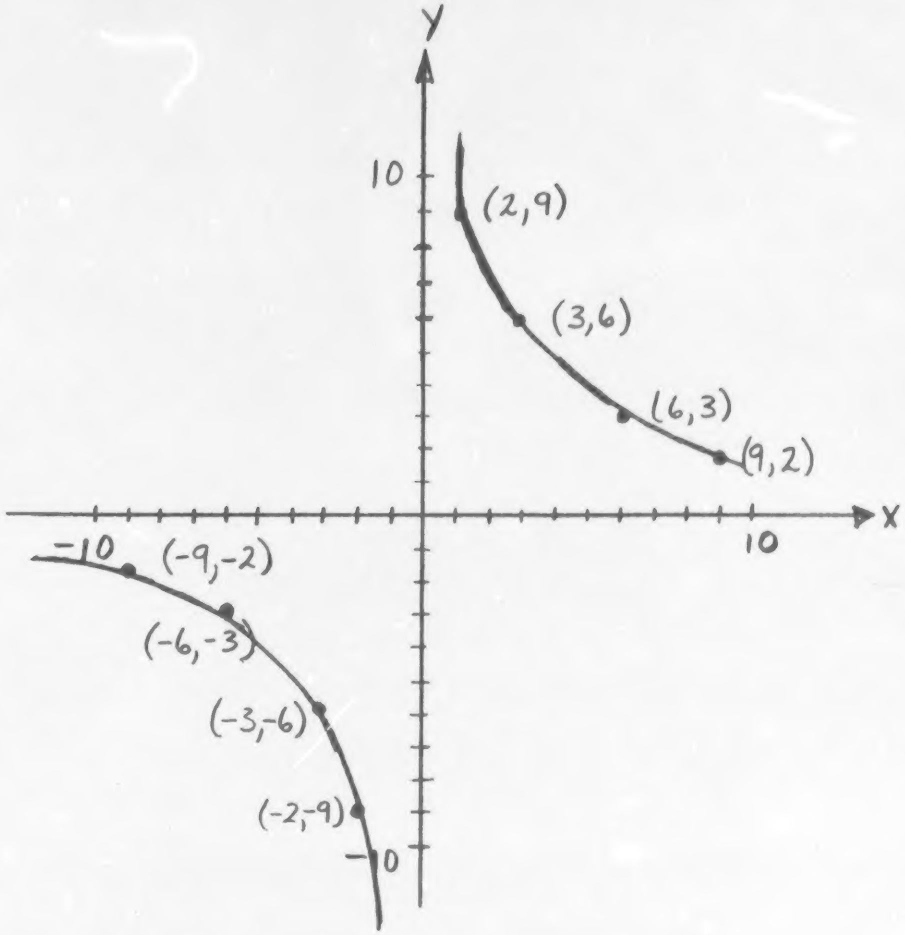
$$x = \frac{18}{y} \quad \text{and} \quad y = \frac{18}{x}$$

Because division by zero is not possible, we cannot allow x or y to assume a value of zero.

Obtaining points for $xy = 18$, by choosing arbitrary values for x and y, are listed in the table:

	$xy = 18$	Points Obtained
If $x = 1$;	$y = 18$	$(1, 18)$
If $x = 2$;	$y = 9$	$(2, 9)$
If $x = 3$;	$y = 6$	$(3, 6)$
If $x = 6$;	$y = 3$	$(6, 3)$
If $x = 9$;	$y = 2$	$(9, 2)$
If $x = -1$;	$y = -18$	$(-1, -18)$
If $x = -2$;	$y = -9$	$(-2, -9)$
If $x = -3$;	$y = -6$	$(-3, -6)$
If $x = -6$;	$y = -3$	$(-6, -3)$
If $x = -9$;	$y = -2$	$(-9, -2)$

Plotting points on axes x and y:



The asymptotes of the equilateral hyperbola, xy = 18, are the x and y axis.

Graph the following conics:

1) $x^2 + y^2 = 9$

2) $2x^2 + 2y^2 = 50$

3) $8x^2 + 4y^2 = 16$

4) $6x^2 + 15y^2 = 30$

5) $y = x^2$

6) $y = x^2 - 8$

7) $x = y^2 - 4$

8) $y = x^2 + 3x - 5$

9) $x^2 - y^2 = 16$

10) $y^2 - x^2 = 25$

THE CONIC SECTIONS

Applications

Circle

EXAMPLE:

A swimming pool of circular shape, with radius = 8 ft, and having constant depth, requires a drain placed on the bottom, where the x coordinate is 5. What is the y coordinate?

SOLUTION:

$x^2 + y^2 = r^2$ Equation of Circle

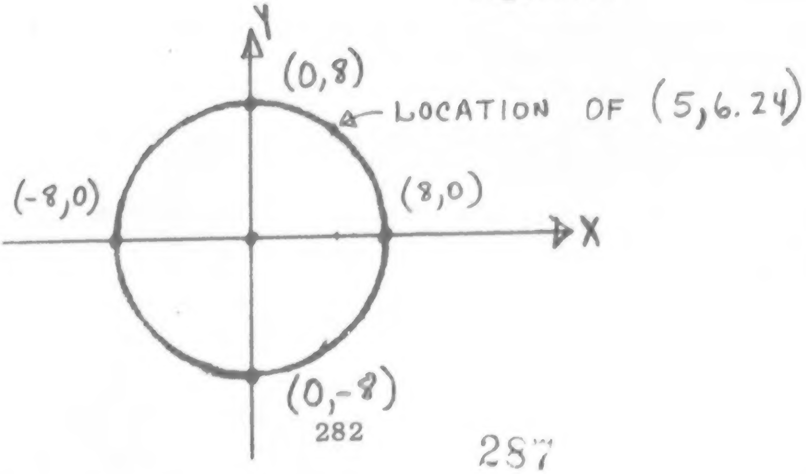
$(5)^2 + y^2 = (8)^2$

$25 + y^2 = 64$

$y^2 = 39$

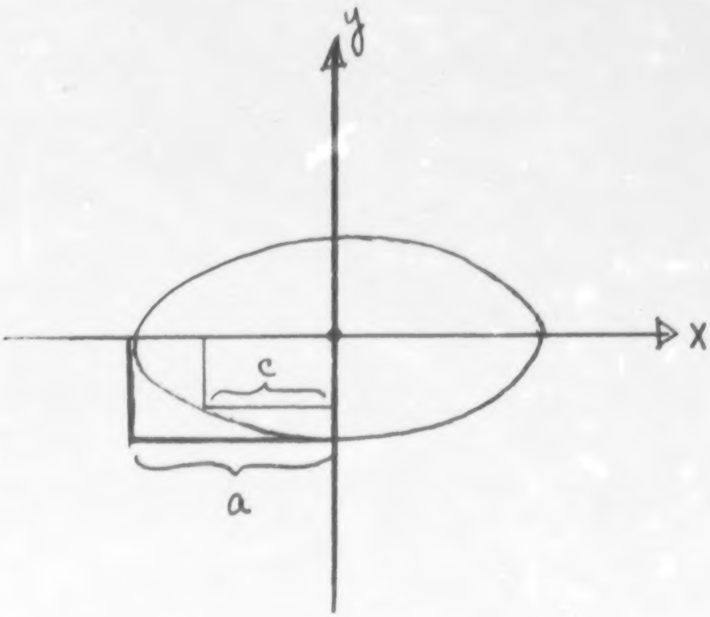
$y = \pm\sqrt{39}$

$y \doteq +6.24$ A length cannot be negative.



Ellipse

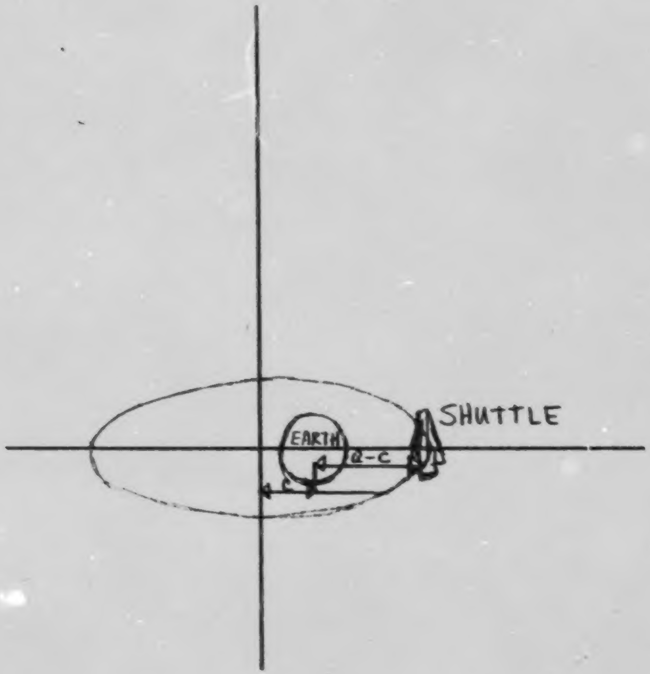
The concept of "eccentricity" has a role in each conic section. By definition, eccentricity, denoted e, is equal to the ratio of the distance from any point on a conic section, to a focus (central point), c, and corresponding directrix (semi-major axis), a. In an ellipse, e < 1. For example:



EXAMPLE:

Imagine the space shuttle travelling in an elliptical orbit with the earth at a mean distance of thirty miles. Eccentricity of the shuttle's orbit if 0.087. What is the nearest and farthest distance of the shuttle from earth?

SOLUTION:



$e = \frac{c}{a} = 0.087$

where a = 30 miles.

From $e = \frac{c}{a}$, conversion will yield:

$c = ea = (0.087)(30 \text{ miles})$

$c = 2.610$

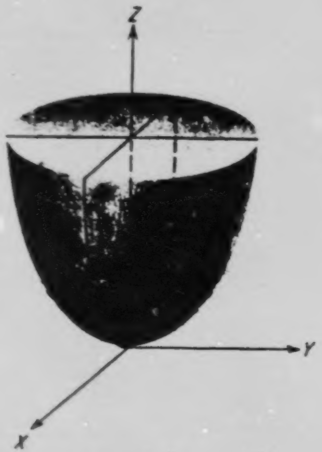
Nearest Distance → 30 miles - 2.61 = 27.39 miles

Farthest Distance → 30 miles + 2.61 = 32.61 miles

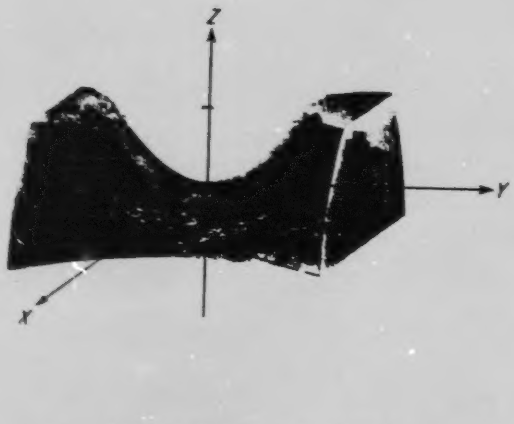
Hyperbola and Parabola

Applications which have algebraic equations resulting in hyperbolas and parabolas, are beyond the scope of this book. However, chemistry, physics, and electronics problems will sometimes involve these figures.

Also, other surfaces are generated from these curves. In three-dimensions, two figures which have a familiar shape occurring in the real world are:



Elliptic Paraboloid



Hyperbolic Paraboloid

From the illustrations, these shapes are generated from the basic curves, parabola and hyperbola. If a parabolic curve was "revolved" about its vertex, another conic is produced, in three-dimensions: an elliptic paraboloid.

By the same technique, revolving a hyperbola will produce a parabolic curve. This figure is known as a hyperbolic paraboloid.

TRIGONOMETRY

Chapter IV - Trigonometry

1. Introduction and Basic Definitions
   - Positive and Negative Angles
2. Trigonometric Functions of:
   - General Angle
   - Acute Angle
3. Trigonometric Functions of: 30 , 45 , 60 , 90
4. Quadrantal Angles
5. Vectors
6. Solutions of Right Triangles
7. Oblique Triangles - Solutions
8. Graphing of Trigonometric Functions
9. Calculators Having Trig Functions
10. Period of Waves
11. Trigonometric Curves - Applications
12. The Sine Curve - Relationship to a Vector

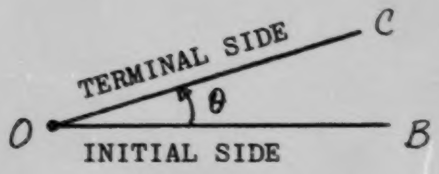
TRIGONOMETRY

An Introduction

In this section, we will study plane trigonometry. The word "plane" means two-dimensional. Trigonometry comes from two Greek words meaning "triangle measure". The study of spherical trigonometry is more advanced, dealing with curved spaces in and about a sphere.

Preparation for trigonometry should include a good background in algebra and a knowledge of geometry concepts. Solving trigonometric equations involves algebra techniques. Laws of exponents, factoring, and solving for unknowns is most important.

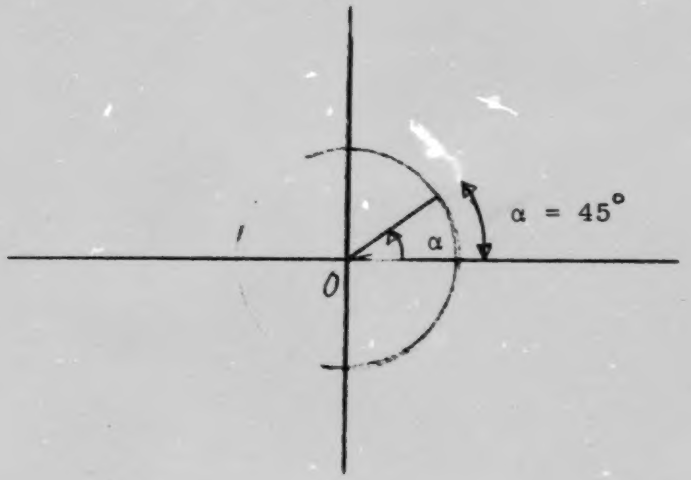
Definition of an Angle



If a line segment, for instance OB, is revolved from a fixed end point, O, to a point C, the angle generated is BOC.

The line segment OB is referred to as the initial side. The line OC is said to be the terminal side. The fixed end point, O, at which the line segment OB revolves, is known as the vertex.

Angles are usually named by Greek letters, such as θ, α, β, γ (theta, alpha, beta, and gamma). Angles are most often measured in degrees.



Suppose there is an angle, α. Using a protractor, α is measured as 45°. What is the size of α in radians, another unit of angle measurement?

A radian is defined to be the angle subtended by a circular arc which is equal in length to the radius of the circle. Conversion factors make finding radians easy:

$$1 \text{ radian} = \frac{180^{\circ}}{\pi}$$

and

$$1 \text{ degree} = \frac{\pi}{180^{\circ}} \text{ radians}$$

If $\alpha = 45^\circ$, therefore, using conversion factors:

$$(45^\circ) \cdot \left(\frac{\pi}{180^\circ}\right) = \frac{45\pi}{180} = \frac{\pi}{4} \text{ radians}$$

If α was first expressed as $\frac{\pi}{4}$ radians, how would the degrees be found?

$$\left(\frac{\pi}{4} \text{ radians}\right) \cdot \left(\frac{180^\circ}{\pi \text{ radians}}\right) = \frac{180^\circ}{4}$$

$$= 45^\circ$$

In addition, make note of the following:

$$\boxed{\begin{aligned} 1 &\doteq 0.017453 \text{ radians} \\ 1 \text{ radian} &\doteq 57.2958^\circ \end{aligned}}$$

Convert the following to radians:

1) 60°

2) 45°

3) 90°

4) 180°

5) 30°

6) 330°

Convert the following to degrees:

1) $\frac{\pi}{2}$

2) $\frac{\pi}{3}$

3) 0

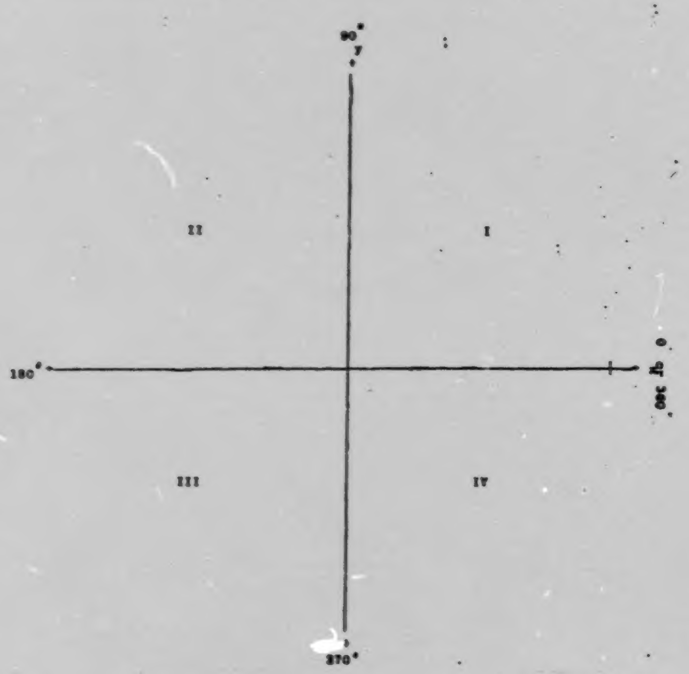
4) $\frac{2\pi}{3}$

5) $\frac{3}{4}\pi$

6) $\frac{5\pi}{6}$

The measurement of angles in degrees is based on sexagesimal measure (base 60). Our number system is base 10, i.e., {0, 1, 2, 3, 4, 5, 6, 7, 8, 9}. Imagine the digits of base 60.

Since we are talking about plane trigonometry, angle measurement takes place on a rectangular coordinate system:



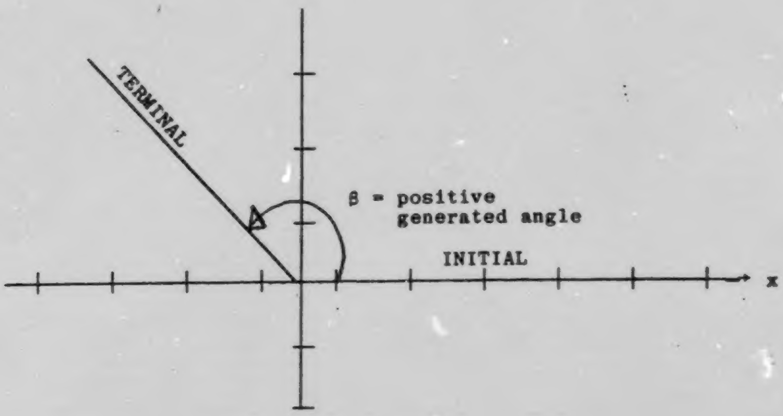
Referring to the diagram, a plane consists of the x and y axes, depicting two-dimensions. The intersection of axes, the origin, is the "vertex". If we consider the positive x axis as the initial side, the x axis is at 0°. Revolving the initial side counter-clockwise, the angle measurement ranges from 0° to 90°. Revolving further, counter-clockwise, the angles measure between 90° and 180°. As the side is revolved more, angles measured are between 180° and 270°, then back to the initial side 0°. However, because a full circle has been rotated, the 0° point is also the 360° mark. The table explains further:

Given an angle, α, on a set of
axes x and y, in standard position:

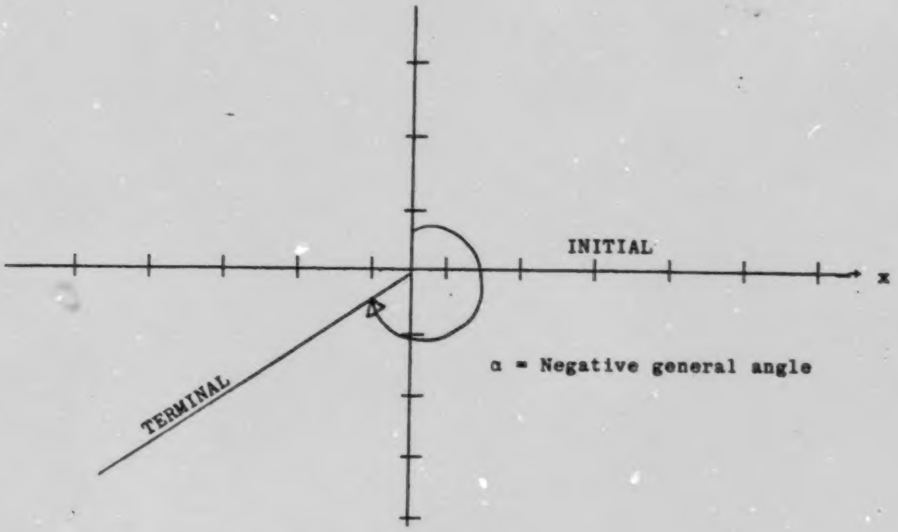
MEASURE	TERM (NAME)
$\alpha = 0^\circ$	straight angle
$0^\circ < \alpha < 90^\circ$	acute angle
$\alpha = 90^\circ$	straight angle
$90^\circ < \alpha < 180^\circ$	obtuse angle
$\alpha = 180^\circ$	straight angle

Negative and Positive Angles

If an angle is generated in a counter-clockwise direction, we have a general angle, which is also positive.



An angle generated by rotating the initial side in a clockwise direction, a negative general angle results.



It is important to note that the definitions of trigonometric functions depends on whether an angle is general or acute. All angles are general, however if the angle falls within the range of $0^\circ < \alpha < 90^\circ$ (acute), the trig functions have definitions based upon the right triangle formed by the initial and terminal side.

Terms of the Sexagesimal System

This base 60 number system is responsible for the following terms utilized in angle degree measurement:

Degree → 1 → $\frac{1}{360}$ of one revolution
Minute → 1' → $\frac{1}{60}$ of one degree
Second → 1" → $\frac{1}{60}$ of one minute

Basic Trigonometric Functions

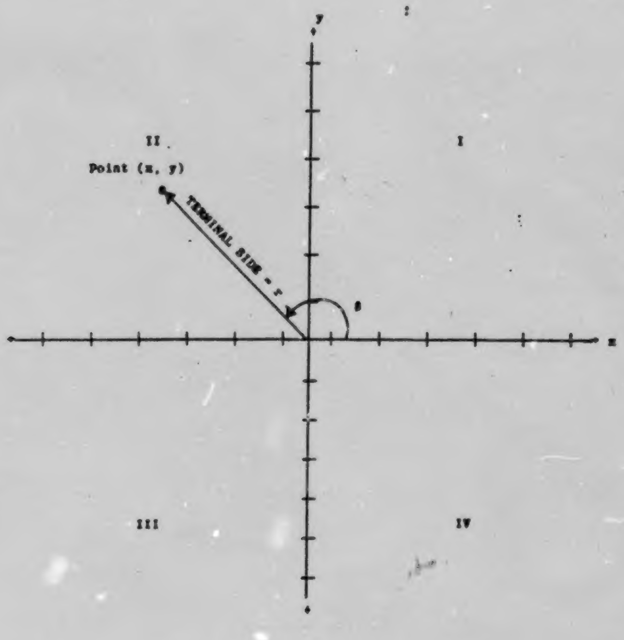
The six basic trigonometric functions are actually ratios of the initial and terminal sides related to a certain angle. Definitions of the functions are dependent upon whether the angle is general, or acute (falling within the range $0^\circ < \alpha < 90^\circ$).

Trig Functions of a General Angle

If angle β is general, and not acute, then α is generated by the terminal side being rotated to a point (x, y) on the rectangular coordinate system. Therefore, the definitions are as follows:

NAME	ABBREVIATION	DEFINITION
sine β	sin β	$\frac{\text{ordinate of point}}{\text{distance from origin}} = \frac{y}{r}$
cosine β	cos β	$\frac{\text{abscissa of point}}{\text{distance from origin}} = \frac{x}{r}$
tangent β	tan β	$\frac{\text{ordinate of point}}{\text{abscissa of point}} = \frac{y}{x}$
cotangent β	cot β	$\frac{\text{abscissa of point}}{\text{ordinate of point}} = \frac{x}{y}$
secant β	sec β	$\frac{\text{distance from origin}}{\text{abscissa of point}} = \frac{r}{x}$
cosecant β	csc β	$\frac{\text{distance from origin}}{\text{ordinate of point}} = \frac{r}{y}$

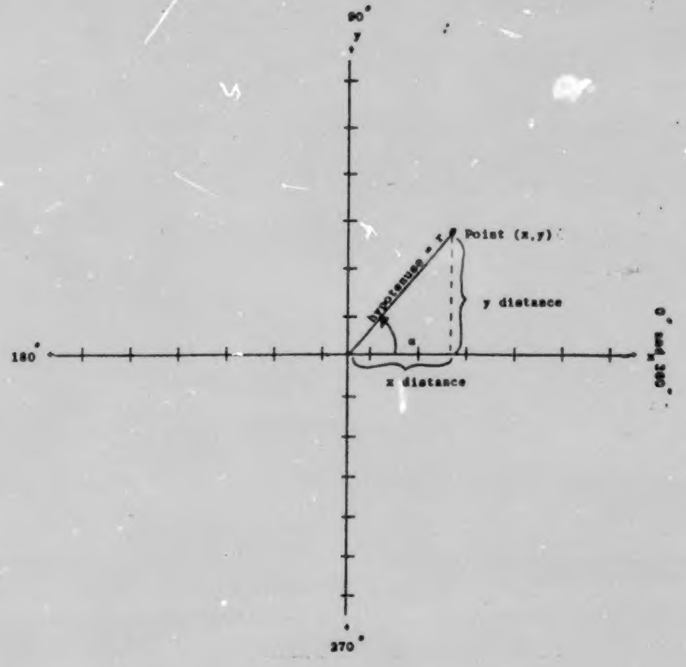
The definitions could be applied pictorially as:



Remember, when dealing with an actual point, the quadrant is important. If the point is in Quadrant II, we have (-x, y). Quadrant III would have points of (-x, -y). And, Quadrant IV contains points (x, -y). Signs of the abscissas and ordinates should be applied in the definitions. The distance from the origin, denoted as r because it is a radius, does not have a negative value at any time.

Trig Functions of an Acute Angle ($0^\circ < \alpha < 90^\circ$)

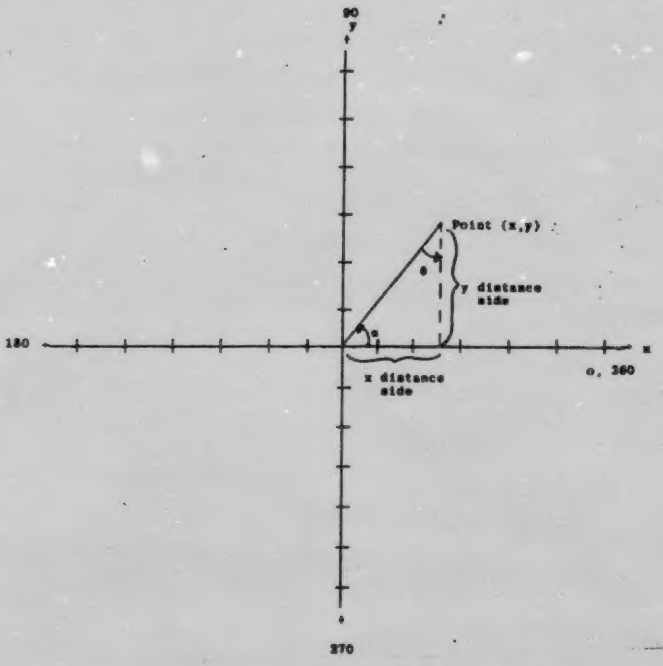
A general angle falling between, but not equal to, 0° → 90°, is called acute.


90°
y
Point (x,y)
hypotenuse = r
y distance
α
180°
x
0° and 360°
x distance
270°

Notice α is a positive, acute angle, formed by the x side and hypotenuse (called r). Therefore, the side adjacent to the angle α is x. The side opposite angle α is y. From this information, we state the definitions of trig functions for an acute angle:

NAME	DEFINITION
$\sin \alpha = \frac{y}{r}$	$\frac{\text{opposite side}}{\text{hypotenuse}}$
$\cos \alpha = \frac{x}{r}$	$\frac{\text{adjacent side}}{\text{hypotenuse}}$
$\tan \alpha = \frac{y}{x}$	$\frac{\text{opposite side}}{\text{adjacent side}}$
$\cot \alpha = \frac{x}{y}$	$\frac{\text{adjacent side}}{\text{opposite side}}$
$\sec \alpha = \frac{r}{x}$	$\frac{\text{hypotenuse}}{\text{adjacent side}}$
$\csc \alpha = \frac{r}{y}$	$\frac{\text{hypotenuse}}{\text{opposite side}}$

If we have another angle, θ, located in the following diagram:



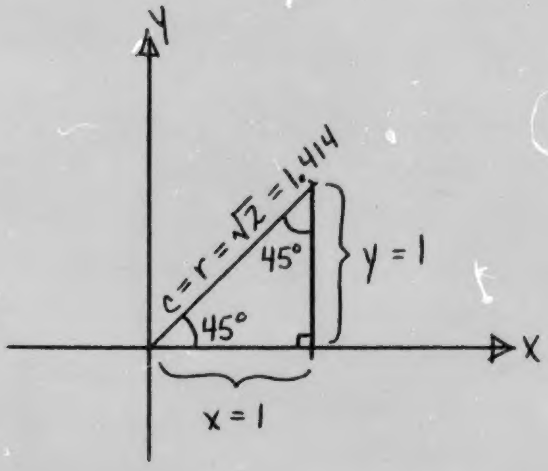
The opposite side of θ would be the x side. And, the adjacent side of θ is the y side. The hypotenuse is never an adjacent side to an angle.

Trigonometric Functions of 30°, 45°, and 60°

These particular angles are used very often. The angle 45 is obtained by constructing a right triangle. Angles 30° and 60° involve an equilateral triangle.

Angle 45°

By definition of a right triangle and the Pythagorean Theorem, we have:



$$c^2 = r^2 = x^2 + y^2$$

$$r = \sqrt{x^2 + y^2}$$

$$r = \sqrt{(1)^2 + (1)^2}$$

$$r = \sqrt{1 + 1}$$

$$c = r = \sqrt{2}$$

Applying the definitions of trig functions, we have:

$$\sin 45^\circ = \frac{y}{r} = \frac{1}{2}$$

$$\cos 45^\circ = \frac{x}{r} = \frac{1}{2}$$

The remaining functions are left to the student.

Angles 30° and 60°

An equilateral triangle is necessary to show angles 30° and 60°. After drawing, a bisector cuts the middle, perpendicular to the base. After the bisector is drawn, a right triangle is formed on both the left and right side. The length of the bisector is found through use of the Pythagorean Theorem.

$$c^2 = r^2 = x^2 + y^2$$

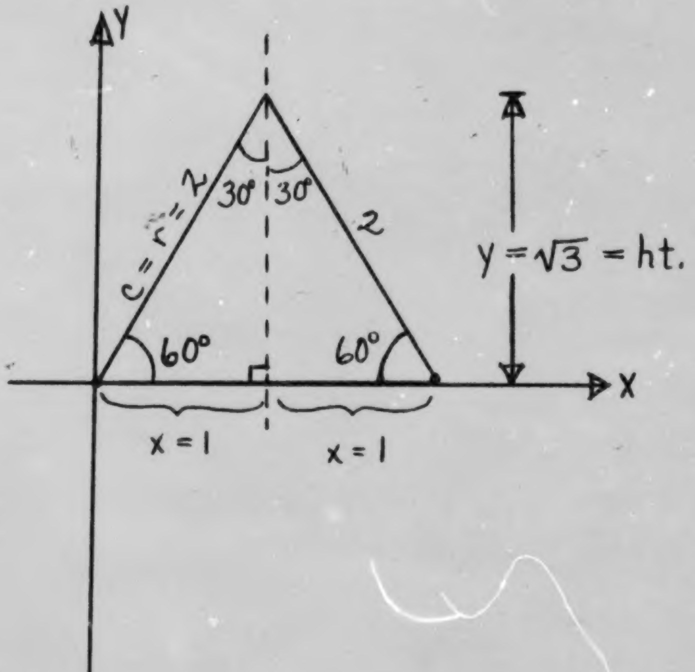
$$r^2 - x^2 = y^2$$

$$\sqrt{r^2 - x^2} = y$$

$$\sqrt{(2)^2 - (1)^2} = y$$

$$\sqrt{4 - 1} = y$$

$$\sqrt{3} = y$$



Example calculations are:

$$\sin 60^\circ = \frac{y}{r} = \frac{\sqrt{3}}{2}$$

$$\cos 60^\circ = \frac{x}{r} = \frac{1}{2}$$

$$\sin 30^\circ = \frac{y}{r} = \frac{\sqrt{3}}{2}$$

$$\cos 30^\circ = \frac{x}{r} = \frac{1}{2}$$

Remaining functions are left to the student.

Trigonometric Functions of Negative Angles

When measuring an angle, the trig functions are dependent upon the values x, y, and r the radius vector. These values have signs according to the quadrant in which an angle is located.

However, if the angle is measured clockwise, a negative angle results. The functions of negative angles are as follows:

$\sin(-\beta) = -\sin\beta$

$\cos(-\beta) = \cos\beta$

$\tan(-\beta) = -\tan\beta$

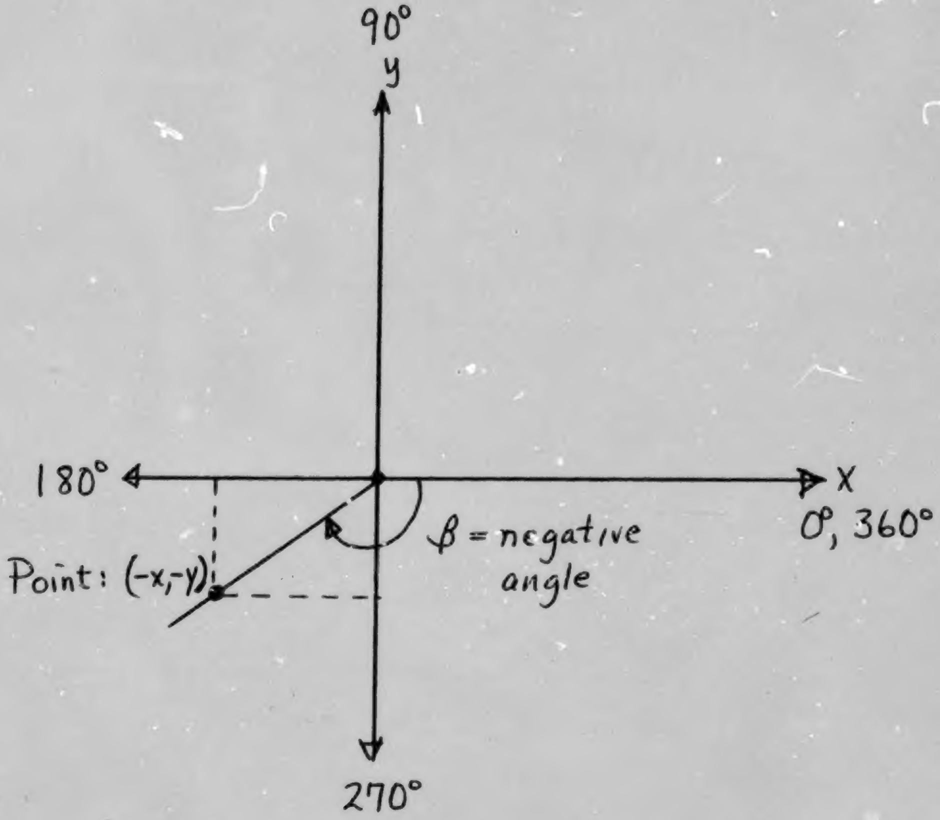
$\cot(-\beta) = -\cot\beta$

$\sec(-\beta) = \sec\beta$

$\csc(-\beta) = -\csc\beta$

We will prove one statement: $\sin(-\beta) = -\sin\beta$.

Consider the following diagram.


90°
y
180°
X
0°, 360°
β = negative angle
Point: (-x,-y)
270°

$$\sin(-\beta) = \boxed{\frac{y}{r}} = \frac{-y}{r} = -\sin\beta$$

The remaining proofs are left to the student.

Prove the following:

1) $\cos(-\beta) = \cos\beta$

2) $\tan(-\beta) = -\tan\beta$

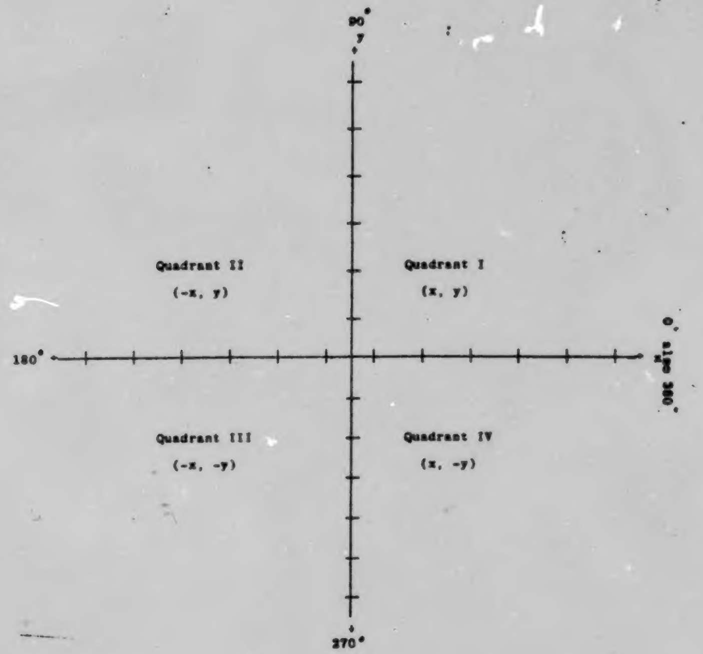
3) $\cot(-\beta) = -\cot\beta$

4) $\sec(-\beta) = \sec\beta$

5) $\csc(-\beta) = -\csc\beta$

Quadrantal Angles

These special angles are called quadrantal because the set of axes x and y, which divide the quadrants of a rectangular coordinate system, compose the five angles.

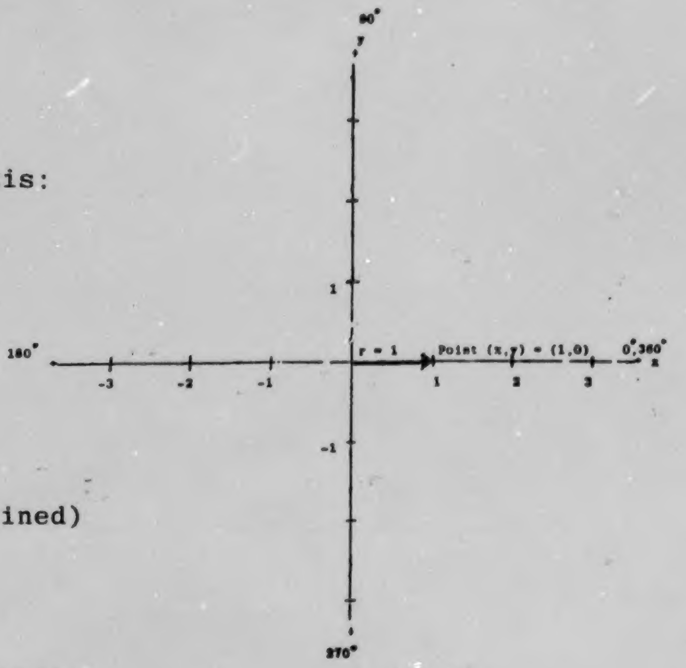

90°
y
Quadrant II
(-x, y)
Quadrant I
(x, y)
180°
x
0° and 360°
Quadrant III
(-x, -y)
Quadrant IV
(x, -y)
270°

The quadrantal angles are: 0°, 90°, 180°, 270°, and 360°.

To find the value of each trig function at these angles, we use a unit vector, or radius. The word unit refers to the length of the vector to a point on an axis; the length of one, also called unit, is the easiest for calculations.

For 0°:

A unit vector, $r = 1$, is drawn to a point (1,0) located on the x axis:



$$\sin 0^\circ = \frac{y}{r} = \frac{0}{1} = 0$$

$$\cos 0^\circ = \frac{x}{r} = \frac{1}{1} = 1$$

$$\tan 0^\circ = \frac{y}{x} = \frac{0}{1} = 0$$

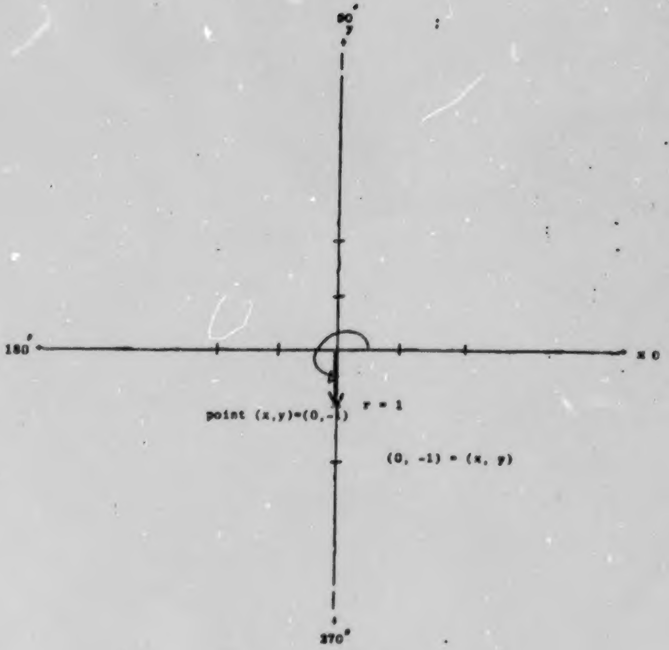
$$\cot 0^\circ = \frac{x}{y} = \frac{1}{0} = \infty \text{ (undefined)}$$

$$\sec 0^\circ = \frac{r}{x} = \frac{1}{1} = 1$$

$$\csc 0^\circ = \frac{r}{y} = \frac{1}{0} = \infty \text{ (undefined)}$$

For 270°:

A unit vector, $r = 1$, is drawn on the negative y axis to a point (0, -1).



$$\sin 270^\circ = \frac{y}{r} = \frac{-1}{1} = -1$$

$$\cos 270^\circ = \frac{x}{r} = \frac{0}{1} = 0$$

$$\tan 270^\circ = \frac{y}{x} = \frac{-1}{0} = \infty$$

$$\cot 270^\circ = \frac{x}{y} = \frac{0}{-1} = 0$$

$$\sec 270^\circ = \frac{r}{x} = \frac{1}{0} = \infty$$

$$\csc 270^\circ = \frac{r}{y} = \frac{1}{-1} = -1$$

Remember, the unit vector, $r = 1$, assumes no negative value at any time.

Can you find the six trig functions of the remaining quadrantal angle?

1) Find values for the six basic trig functions for general angle β having initial side on the x - axis and terminal side ending at the designated point.

a. (-2, 3)

b. (1, -2)

c. (-3, -3)

d. (-1, 5)

e. (3, -1)

2) Find values for the six basic trig functions for acute angle α whose initial side lies on the x - axis and terminal side ends at the designated point.

a. (2, 0)

b. (4, 3)

c. (0, 5)

d. (1, 1)

e. (3, 5)

3) Find the ratio of r : x for the angles -

a. 30°

b. 45°

c. 60°

4) Find values of tan, cot, sec, and csc for angles -

a. 30°

b. 45°

c. 60°

Vectors

When discussing physical quantities, such as force, velocity, and torque (the tendency of a force to produce rotation about an axis), a line segment is drawn on paper to represent the quantity.

These line segments are called VECTORS, only if the physical quantics they represent possess the following characteristics:

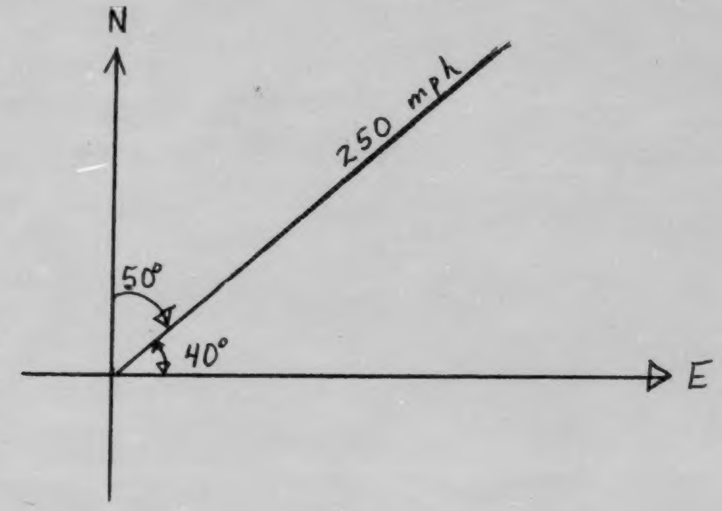
1) A point of application, usually the particle or object itself.

2) Magnitude, which is the size or greatness of the quantity, usually described by units of pounds, dynes, Newtons, etc.

3) Direction, indicated by an arrowhead. Also, a "line of action", drawn through the point of application and in the direction of the arrowhead.

EXAMPLE

An airplane traveling at 250 mph in a direction 50° NE, is denoted mathematically (by vectors) as:



Therefore, the velocity of the plane can be represented as a vector, because there is:

1) A point of application = the airplane

2) Magnitude = 250 mph.

3) Direction = 50 NE.

Scalers

A scaler is a physical quantity which does not have all the characteristics of a vector.

Examples of scaler quantities are time and energy. These quantities possess magnitude, but not direction.

In the preceding example, velocity was expressed as a vector. Usually, velocity is a scaler quantity. The example did state a direction of 50° NE as well as the magnitude 250 mph. Therefore, in this case, velocity can be expressed as a vector quantity.

Addition and Subtraction of Vectors

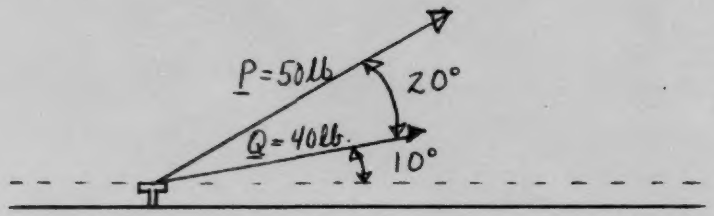
Vectors cannot be added or subtracted in the manner we know. However, scaler quantities can be added in the usual way.

Commutative and Associative properties apply to physical quantities. Therefore, the properties apply to vectors.

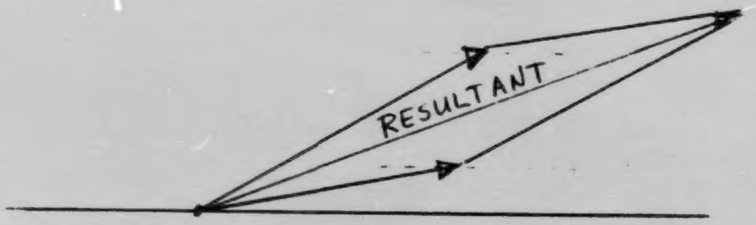
To add two vectors, we use a method called the PARALLELOGRAM LAW.

Example

Suppose there are two forces, P and Q, acting on a nail head.



Use the parallelogram law to add vectors P and Q.



A dotted line, the same length, direction, and angle of elevation as Q, is placed beginning at the end point of P.

A dotted line, the same length, direction, and angle of elevation as P, is placed beginning at the end point of Q.

Next, draw a diagonal from the point of application, A, which is the nail head, to the point connecting the dotted lines.

This diagonal is called the RESULTANT. It is the result of adding P and Q.

The lengths of P and Q aid in determining a numerical value of the RESULTANT. The length of the resultant is about $1\frac{2}{3}$ greater the length of vector P. If P represents 50 lb, then:

$$\text{Resultant} = 50 \text{ lb} \cdot 1\frac{2}{3}$$

$$\underline{R} = \frac{50 \text{ lb}}{1} \cdot \frac{5}{3}$$

$$\underline{R} = \frac{250 \text{ lb}}{3}$$

$$\underline{R} = 83.3 \text{ lb.}$$

Also, the angle of elevation of R is part of the answer. Using a common protractor, the angle is 20°. The final answer is expressed:

$$\underline{R} = 83.3 \text{ lb, angle} = 20^\circ$$

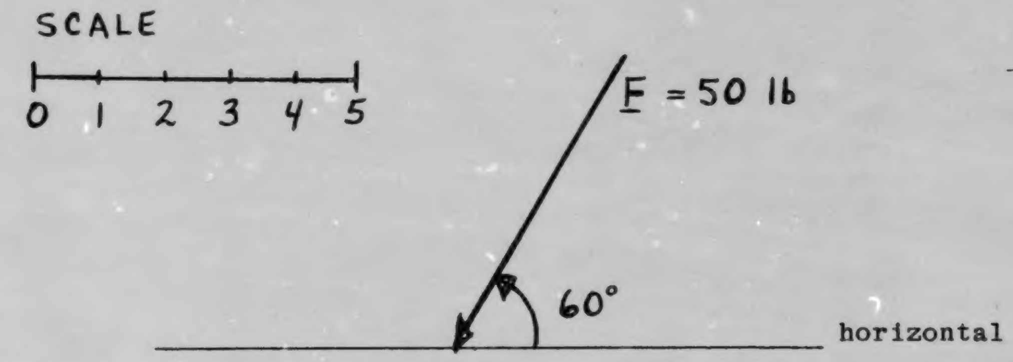
NOTE:

Another diagonal can be drawn in the figure. However, this diagonal would not yield a correct, resultant force.

The Resultant, R, is the equivalent of the two forces P and Q acting on the same point of application.

Horizontal and Vertical Components of a Force Vector

Suppose there is a force, $F = 50$ lbs, applied downward at an angle of 60° with the horizontal. By illustration:


SCALE
0
1
2
3
4
5
$\underline{F} = 50$ lb
60°
horizontal

To determine horizontal and vertical components of the force, $\underline{F}$, use:

$$\underline{F}_{\text{HORIZONTAL}} = \underline{F}\,(\cos 60^\circ) \qquad \text{RULE}$$

$$= 50 \text{ lbs } (0.5)$$

$$= 25 \text{ lbs}$$

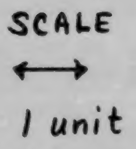
$$\underline{F}_{\text{VERTICAL}} = \underline{F}\,(\sin 60^\circ) \qquad \text{RULE}$$

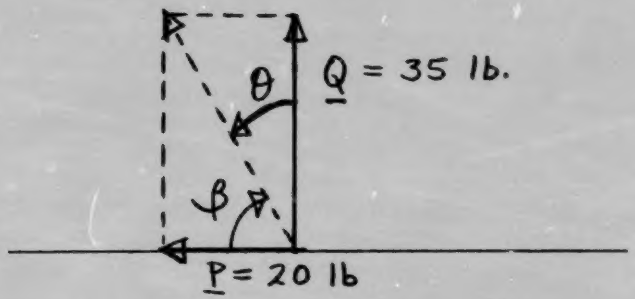
$$= 50 \text{ lbs } (.866)$$

$$\doteq 43.3 \text{ lb}$$

EXAMPLE:

If a force, P = 20 lbs is acting in a horizontal direction, with a force Q = 35 lbs acting vertically, what is the resultant of the two forces?





By constructing broken lines, we can see a right triangle is formed, with side c, or hypotenuse, the resultant force to be found.

Using the Pythagorean Theorem, the magnitude (or greatness) of the resultant, R, can be found.

Q → base of triangle = 35 = b

P → side a of triangle = 20 = a

R → c of triangle = ?

Solving:

$$c^2 = a^2 + b^2$$

$$c = \sqrt{a^2 + b^2}$$

$$c = \sqrt{(20)^2 + (35)^2}$$

$$c = \sqrt{1625}$$

$$c \doteq 40.3 \rightarrow R, \text{ the resultant force}$$

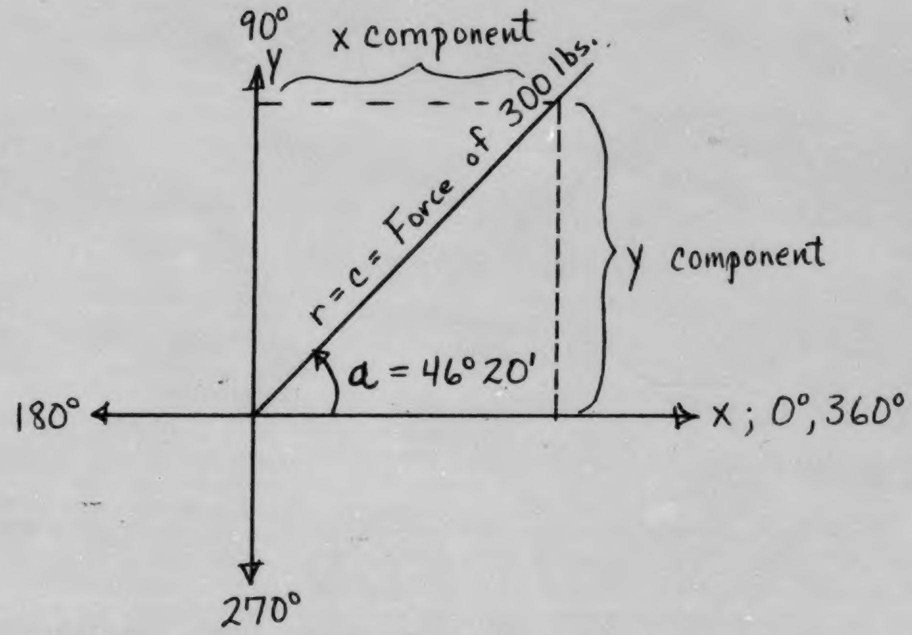
The direction of R, measured in degrees, is obtained by:

From the Vertical	From the Horizontal
$\tan \beta = \frac{y}{x}$	$\theta = 90^\circ - \beta$
$\tan \beta = \frac{35}{20}$	$\theta = 90^\circ - 60.2^\circ$
$\tan \beta = 1.75$	$\theta = 29.8^\circ$
$\beta = 60.2^\circ$	

EXAMPLE:

Suppose a force of 300 lbs. is acting an angle of 46° 20' with the horizontal. What are the horizontal and vertical components?

Solution: Remembering from vectors, we construct a diagram, labeling all parts:


90°
y
x component
r = c = Force of 300 lbs.
y component
a = 46° 20'
180°
x ; 0°, 360°
270°

From the definition of basic trig functions:

$$\sin \alpha = \frac{y}{r}, \text{ then } \sin 46^\circ 20' = \frac{y}{300 \text{ lbs.}}$$

$$(300 \text{ lbs.})(\sin 46^\circ 20') = y$$

$$(300 \text{ lbs.})(0.7234) = y$$

$$217 \text{ lbs.} = y$$

Also, from the definition of trig functions:

$$\cos \alpha = \frac{x}{r}, \text{ then } \cos 46^\circ 20' = \frac{x}{300 \text{ lbs.}}$$

$$(300 \text{ lbs.})(\cos 46^\circ 20') = x$$

$$(300 \text{ lbs.})(0.6905) = x$$

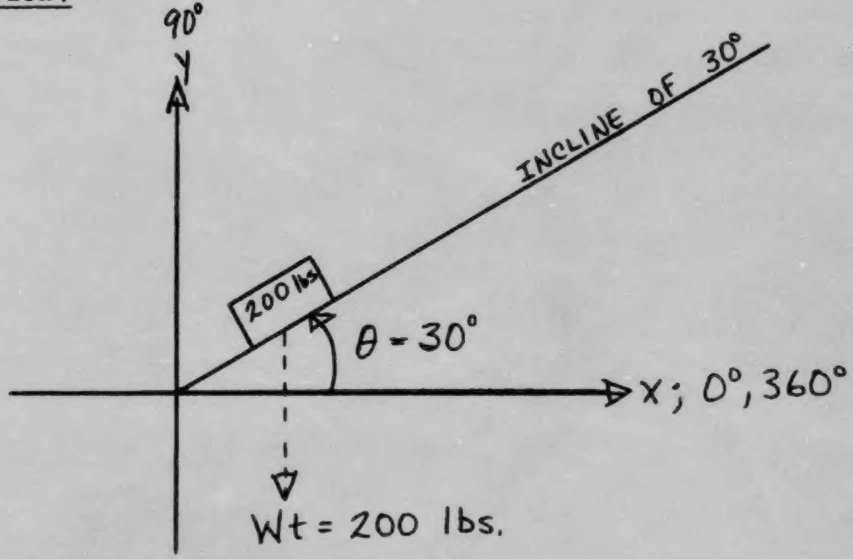
$$207 \text{ lbs} \doteq x$$

Therefore, the vertical component of a resultant force of 300 lbs. is approximately 217 lbs. The horizontal component, x, is about 207 lbs.

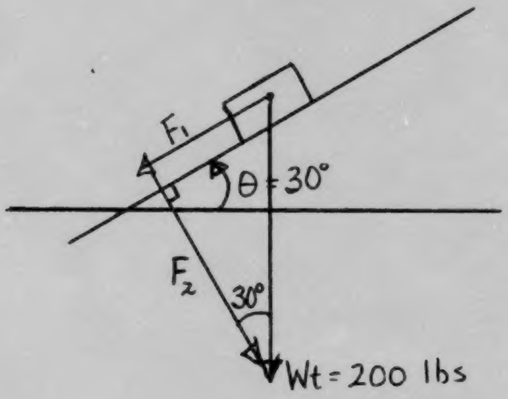
EXAMPLE:

A block weighing, wt = 200 lbs, is at rest on an incline of 30° with the horizontal. What is the force of the block sitting on the incline?

Solution:



After drawing a diagram illustrating the block and incline, draw another illustration of just the vectors involved; i.e., resolve the weight of the block into its two components of force:



Tip to tail method of vectors creates a right triangle.

F_1 → base; opposite

F_2 → adjacent

wt → hypotenuse

F_2 → Force of the block on the ramp.

To find the value of F_2,

$$\cos 30^\circ = \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{F_2}{\text{wt}}$$

$$\cos 30^\circ = \frac{F_2}{200 \text{ lbs.}}$$

$$(200 \text{ lbs})(\cos 30^\circ) = F_2$$

$$(200 \text{ lbs})(0.8660) = F_2$$

$$173.2 \text{ lbs} = F_2$$

Therefore, a force of 173.2 lbs is acting on the block as it rests on a 30° inclined ramp.

Solutions of Right Triangles

In order to solve for a missing part of a right triangle, we must know:

$$\boxed{\alpha + \beta = 90^\circ}$$

and, either have a calculator, or a table of natural trigonometric functions.

EXAMPLE:

Given: A right triangle having an angle $\alpha = 37^\circ\ 23'$, and side $x = 1.375$.

Solution: In a right triangle,

$$\alpha + \beta = 90^\circ$$

$$37^\circ\ 23' + \beta = 90^\circ$$

$$\beta = 90^\circ - 37^\circ\ 23'$$

$$\beta = 52^\circ\ 37'$$

We need to know the opposite side, y, of angle α. Values known at this point are: α, β, adjacent side x. Therefore, we use a trig function definition to find y:

$$\tan \alpha = \frac{\text{opposite}}{\text{adjacent}}$$

$$\tan 37^{\circ}\,23' = \frac{y}{1.375}$$

$$(1.375)(\tan 37^{\circ}\,23') = y$$

$$(1.375)(0.7627) = y$$

$$1.0487 = y$$

The hypotenuse c is found by use of the Pythagorean Theorem:

$$r^2 = c^2 = x^2 + y^2$$

$$c = \sqrt{x^2 + y^2}$$

$$c = \sqrt{(1.375)^2 + (1.0487)^2}$$

$$c = \sqrt{1.89 + 1.099}$$

$$c = \sqrt{2.9897}$$

$$c = 1.729$$

Solutions to the problem are:

$\alpha = 37^{\circ}\,23'$, $\beta = 52^{\circ}\,37'$, $x = 1.375$,

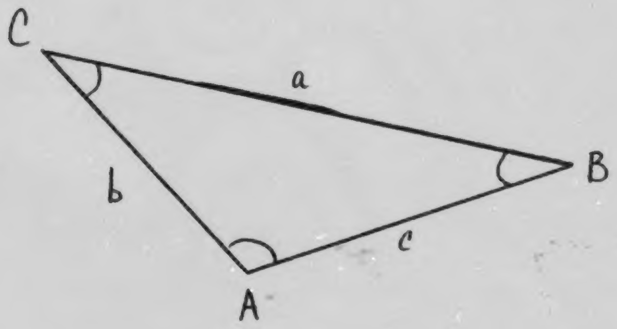
$y = 1.0487$, $r = c = 1.729$

The decimal value of minutes was found by dividing 60 into 35 $\rightarrow \frac{35}{60}$, because 60 minutes are in one degree. Therefore, punching in 28.58333^{c} will yield a value for any trig function.

Oblique Triangles - Solutions

It has been shown that right angles and right triangles occur quite frequently in trigonometry. However, problems in nature do not exist only as right triangles. In this section, we investigate oblique triangles.

By definition, an oblique triangle is a triangle that does not contain a right, or 90°, angle. This particular triangle could look like the following:



with A, B, C representing angles, and lower-case letters a, b, c denoting sides.

To solve any oblique triangle, an important rule can be used:

Law of Sines

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

The Law of Sines is defined as having "sides proportional to the sines of the opposite angles".

Depending upon what is known and what quantity is being solved for, we use the Law of Sines in the following manner:

$$\frac{a}{\sin A} = \frac{b}{\sin B}$$

or

$$\frac{a}{\sin A} = \frac{c}{\sin C}$$

or

$$\frac{b}{\sin B} = \frac{c}{\sin C}$$

When solving right triangles, the geometric statement $\alpha + \beta = 90^c$ was helpful in determining a missing angle. In oblique triangles, the following statement can be used:

$$\boxed{\begin{array}{c}\alpha + \beta + \gamma = 180^c \\ \text{or} \\ A + B + C = 180^c\end{array}}$$

Before attempting to solve an oblique triangle, one of the following conditions must be met:

I. A side length and two angles are known.

II. Two side lengths and the angle opposite one side should be given.

III. Two side lengths and the included angle are defined.

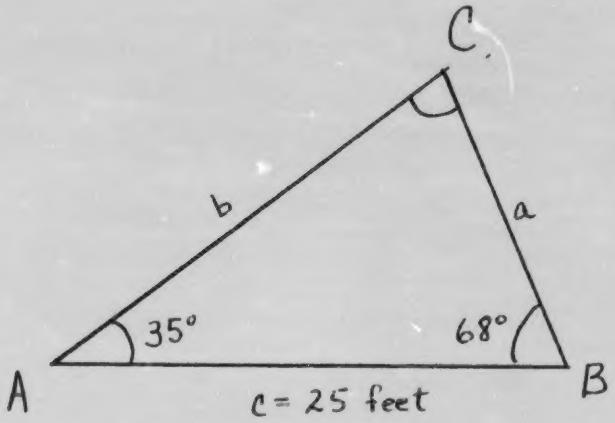
IV. The lengths of all three sides are known.

Therefore, if we are given information to help solve an oblique triangle, the information provided must include one of the above listed conditions.

EXAMPLE:

A plot of land is marked off as a triangle due to natural springs on the property. The surveyor has measured two angles, A and B, as 35° and 68°, respectively. One side length, called c, is 25 feet. What is the measurement of the third angle, and the length of sides a and b?

SOLUTION:



First, we construct a diagram and list the known (given) quantities. We know two angles and one side length. This satisfies condition I, meaning we can now apply the law of sine, and solve for missing dimensions.

We also know the triangle is oblique. Because:

$$A + B + C = 180^\circ$$

$$35^\circ + 68^\circ + C = 180^\circ$$

$$103^\circ + C = 180^\circ$$

$$C = 180^\circ - 103^\circ$$

$$C = 77^\circ$$

Neither A, B, or C is a right (90°) angle.

To find the missing sides, b and a, apply the Law of Sines:

$$\frac{a}{\sin A} = \frac{c}{\sin C}$$

$$\frac{a}{\sin 35} = \frac{25\text{ ft.}}{\sin 77}$$

$$a(\sin 77^\circ) = 25\text{ ft }(\sin 35^\circ) \qquad \text{means = extremes}$$

$$a(0.974) = 25\text{ ft }(0.573)$$

$$\frac{a(0.974)}{0.974} = \frac{25\text{ ft}(0.573}{0.974}$$

$$a = \frac{14.325\text{ ft}}{0.974}$$

$$a = 14.707\text{ ft.}$$

Now, to find side b:

$$\frac{a}{\sin A} = \frac{b}{\sin B}$$

$$\frac{14.707 \text{ ft}}{\sin 35^\circ} = \frac{b}{\sin 68^\circ}$$

$$(14.707 \text{ ft})(\sin 68^\circ) = b(\sin 35^\circ) \qquad \text{means = extremes}$$

$$(14.707 \text{ ft})(0.927) = b(0.573)$$

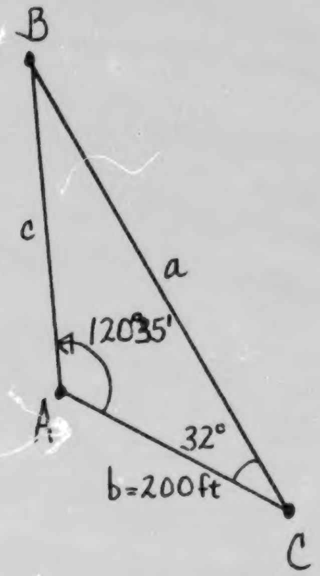
$$\frac{13.633 \text{ ft}}{0.573} = b$$

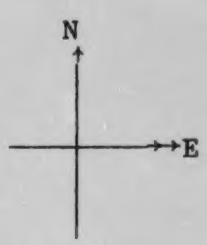
$$23.792 \text{ ft} = b$$

EXAMPLE:

Suppose there are two houses, A and B, on opposite sides of a river. From house A, the boat dock is located 200 ft., northeast at a point C. The angles of A and C are 120° 35' and 32°, respectively. What is the distance between house A and B? And also solve the distance between house B and the boat dock, C.

SOLUTION:





To find the distance of AB, which is side length c, use law of sines:

$$\frac{b}{\sin B} = \frac{c}{\sin C}$$

$$\frac{200 \text{ ft}}{\sin B} = \frac{c}{\sin 32}$$

We cannot proceed because there are two unknowns, c and angle B. First, find B:

$$A + B + C = 180^\circ$$

$$120^\circ\ 35' + B + 32^\circ = 180^\circ$$

$$B + 152^\circ\ 35' = 180^\circ$$

$$B = 27^\circ\ 25'$$

Now proceed with Law of Sines:

$$\frac{200\text{ ft}}{\sin 27^\circ\ 25'} = \frac{c}{\sin 32^\circ}$$

$$c(\sin 27^\circ\ 25') = (200\text{ ft})(\sin 32^\circ)$$

$$c(0.460) = (200\text{ ft})(0.529)$$

$$c(0.460) = 105.8\text{ feet}$$

$$c = \frac{105.8\text{ ft}}{0.460}$$

$$c = 230\text{ ft}$$

To find the distance between house B and the boat dock C, side length a, use Law of Sines:

$$\frac{a}{\sin A} = \frac{b}{\sin B}$$

$$\frac{a}{\sin 120^\circ\ 35'} = \frac{230\text{ ft}}{\sin 27^\circ\ 25'}$$

$$a(\sin 27^\circ\ 25') = (230\text{ ft})(\sin 120^\circ\ 35')$$

$$a(0.460) = (230\text{ ft})(0.860)$$

$$a = 430.4\text{ ft.}$$

1) Use the law of sines to solve the triangles with the following components.

a. $a = 2$, $A = 52^\circ$, $B = 70^\circ$

b. $a = 3$, $b = 5$, $A = 41^\circ$

c. $a = 1$, $b = 3$, $B = 5^\circ$

Graphing Trigonometric Functions

We have learned how to find the value of trigonometric functions in terms of x, y, and the hypotenuse (radius vector). These functions were defined as "ratios" of line distances which comprise a particular angle, on a rectangular coordinate system.

In order to graph trigonometric functions, they must be in the form of:

$$\boxed{y = \text{trigonometric function}}$$

Remembering linear equations, before graphing, the equation must be in the form $y = mx + b$. Therefore, trigonometric functions should be in the form of y = trigonometric function of some x.

Investigating the sine function of some angle α, if placed in the proper form for graphing, we would have:

$$y = \sin x$$

with x representing some angle α. Thus, for every angle x, there is a corresponding value for y. Explaining further:

If $x = 0^\circ \rightarrow y = \sin x$

$y = \sin 0^\circ$ Point: $(0^\circ, 0)$

$y = 0$

If $x = 30^\circ \rightarrow y = \sin x$

$y = \sin 30^\circ$ Point: $(30^\circ, .5)$

$y = 0.5$

As you can see, for every value of x in degrees, we have a corresponding value of y, which is not in degrees. Investigating angles from 0° to 360°, in increments of 30°, the following values would result:

If x =	y = sin x	y =
0°	y = sin 0°	0.000
30°	y = sin 30°	0.500
60°	y = sin 60°	0.866
90°	y = sin 90°	1.000
120°	y = sin 120°	0.866
150°	y = sin 150°	0.500
180°	y = sin 180°	0.000
210°	y = sin 210°	-0.500
240°	y = sin 240°	-0.866
270°	y = sin 270°	-1.000
300°	y = sin 300°	-0.866
330°	y = sin 330°	-0.500
360°	y = sin 360°	0.000

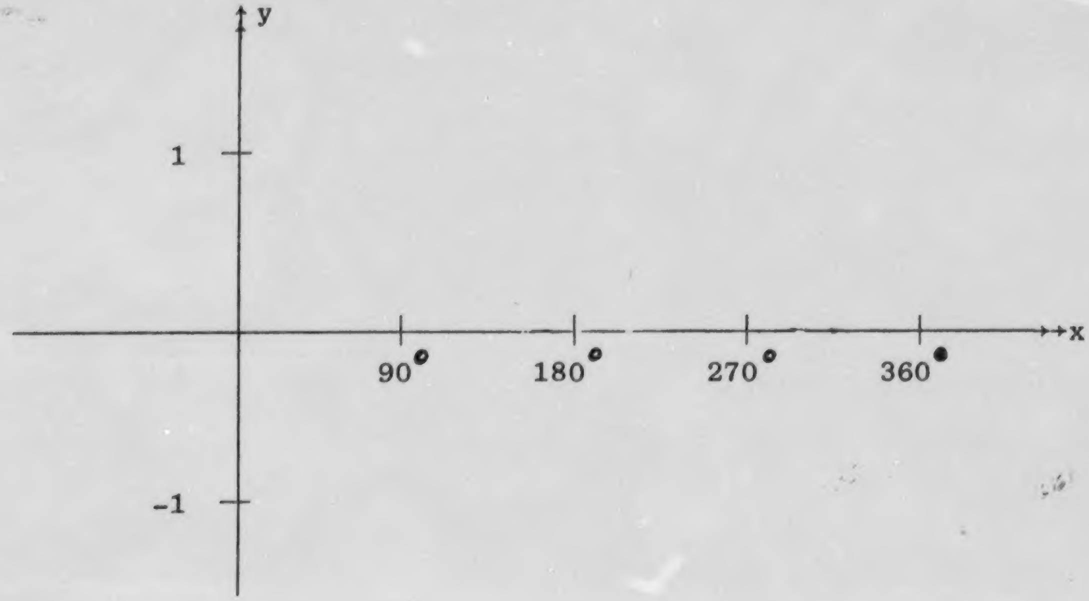
Referring to the table, notice that as the angle x increases from 0° to 90°, the value of y increases from 0.000 to 1.000.

As the angle x continues to increase, from 90° to 180°, the y values start to decrease: from 1.000 back to 0.000.

At the points where the angle x increases from 180° to 270°, the values of y are negative, which means the y values are getting smaller (or decreasing).

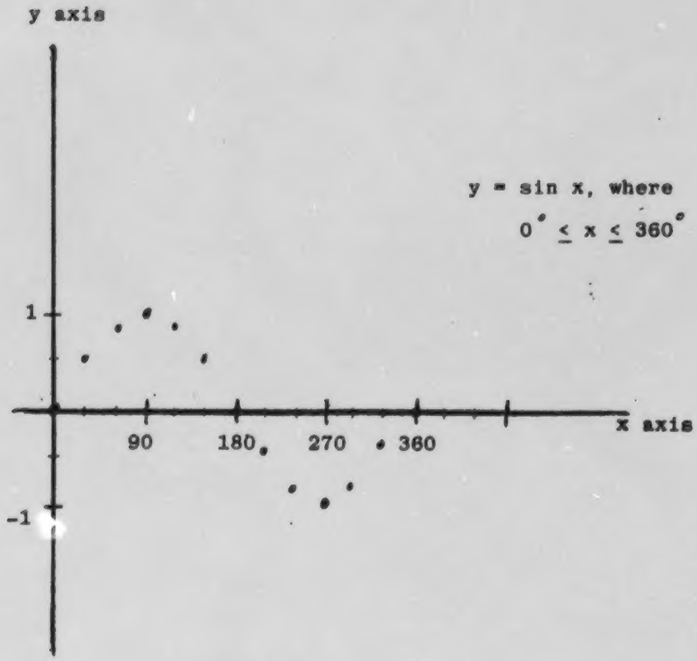
The remaining angles x, from 270° to 360° (full circle), the y values remain negative till y = 0, however from -1.000 to 0.000 the y values are increasing.

To depict these degree values of x and the varying numerical values of y, we use a graph such as:



Notice in this graph, the y axis consists of the point (0, 1) and (0, -1). The x axis has degree values, in increments of 90°, through 360°.

In order to graph the function y = sin x, refer to the table containing the points obtained as x increases from 0° to 360°.



The curve resulting from plotting points, is called the basic sine curve. The curve peaks at y = 1 when x = 90°. As the curve goes downward, it crosses the x axis at x = 180°, because y = 0. After peaking at y = -1 when x = 270°, the curve continues upward again, crossing the x axis at x = 360°, with y = 0.

Period of Waves

When graphing trigonometric functions, the resulting curves have a recurring form, which are repeated at regular intervals; i.e., from one degree value to another degree value.

The sine curve "peaks" at a positive y value (y = 1), then decreases to a "trough" at a negative y value (y = -1, before crossing the x axis, again, in an upward motion. The completion of a "positive peak" and a "negative trough" results into what is called a period.

As we continue graphing the functions of y = cos x, y = tan x, etc., we will notice that the resulting curves have a period, as did the sine curve. The period of each trigonometric function is found to be 360°. Explaining further, this means that each curve will have a "positive peak" and a "negative trough", every 360°.

> A trigonometric function is said to have a curve that is "periodic", if the two peaks are repeated at regular intervals, called "periods".

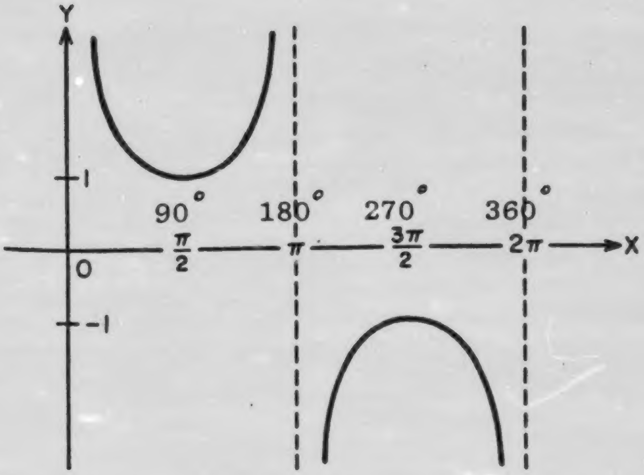
Our definition of periodicity is rather loosely defined. However, please consult a trigonometry book, or guide, for more complete explanations.

Graphs of Trigonometric Functions Containing $y = \infty$ (Undefined)

In the previous section, the graphs of the basic trig functions were illustrated. These curves can be checked for accuracy by substituting degree values for x, and then solving for y, in each function.

By investigating the curves, the graphs of y = tan x, y = cot x, y = sec x, and y = csc x, look a little strange. They also contain "broken" lines at certain points. We will explain in this section, why the curves have a strange appearance.

Actually, the reasons are very simple. By substituting degree value for x, we obtain numerical values for y. Refer to the next graph:



y = csc x

The curve of $y = \csc x$ has a y value of 1 when $x = 90^\circ$. Why? Notice:

$$\text{If } x = 90^\circ; \quad y = \csc x$$

$$y = \csc 90^\circ$$

$$y = \frac{1}{\sin 90^\circ} \qquad \text{Remember } \csc x = \frac{1}{\sin x}$$

$$y = \frac{1}{1} = 1$$

Looking at the graph of $y = \csc x$, we notice that a broken-line occurs at $x = 180$. Why? By substituting $x = 180$ into the function, we have:

$$\text{If } x = 180^\circ; \quad y = \csc x$$

$$y = \csc 180^\circ$$

$$y = \frac{1}{\sin 180^\circ}$$

$$y = \frac{1}{0}$$

$y = \infty \rightarrow$ undefined, because division by zero is not possible.

Therefore, since a y value is undefined, or not possible at $x = 180^\circ$, a broken line is drawn on the graph. This line depicts that the curve resulting from $y = \csc x$ cannot take on a y value at $x = 180^\circ$. In a sense, the broken line serves as a reminder that the curve cannot cross or assume any y value on the graph at x = 180°. This occurs again at x = 360°. Why?

$$\text{If } x = 360^\circ; \quad y = \csc x$$

$$y = \csc 360^\circ$$

$$y = \frac{1}{\sin 360^\circ}$$

$$y = \frac{1}{0}$$

$$y = \infty \rightarrow \text{undefined}$$

Calculators Having Trigonometric Functions

Usually, a good calculator can be bought for an inexpensive price. Most calculators designed for classroom use do have trigonometric functions. However, often the "common" functions only are included: sine, cosine, and tangent.

In solving trigonometric equations, it is sometimes necessary to utilize the functions which are not common: cotangent, secant, and cosecant.

These "uncommon" functions are actually reciprocals of the "common" trigonometric functions.

$$\boxed{\begin{array}{l} \text{cotangent } x \rightarrow \cot x = \dfrac{1}{\tan x} \\[2ex] \text{secant } x \rightarrow \sec x = \dfrac{1}{\cos x} \\[2ex] \text{cosecant } x \rightarrow \csc x = \dfrac{1}{\sin x} \end{array}}$$

Knowing the relationships existing between the functions, it is easy to solve for function values of the "uncommon" ratios.

EXAMPLE:

Find: $\sec 180^\circ$

Solution:

$$\sec 180^\circ = \frac{1}{\cos 180^\circ}$$

$$= \frac{1}{-1}$$

$$= -1$$

EXAMPLE:

Find: $\cot 180^\circ$

Solution:

$$\cot 180^\circ = \frac{1}{\tan 180^\circ}$$

$$= \frac{1}{0}$$

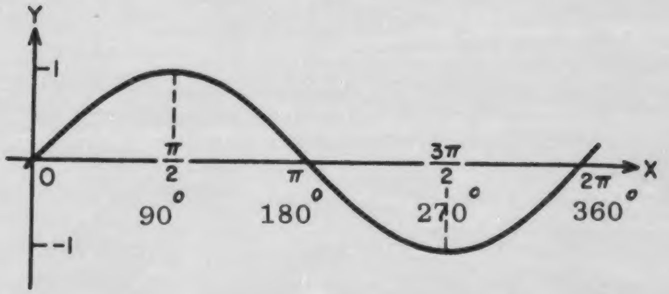
$$= \infty \rightarrow \text{undefined}$$

Though hand calculators may be programmed for trig functions in degrees, how would the calculator evaluate an angle, for instance, $28^\circ 35'$?

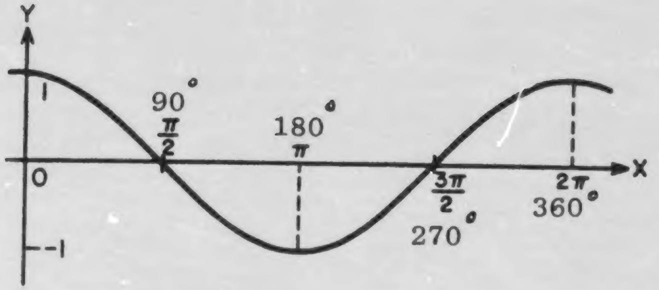
To "punch in" and evaluate an angle given in degrees and minutes, translate the minutes portion into a decimal value, For example:

$$28^\circ\ 35' = 28.58333^\circ$$

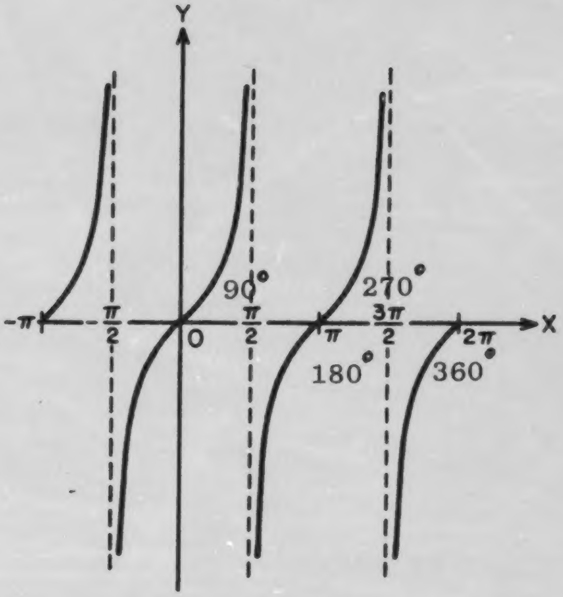
The following graphs depict the resulting curves of the six basic trigonometric functions:



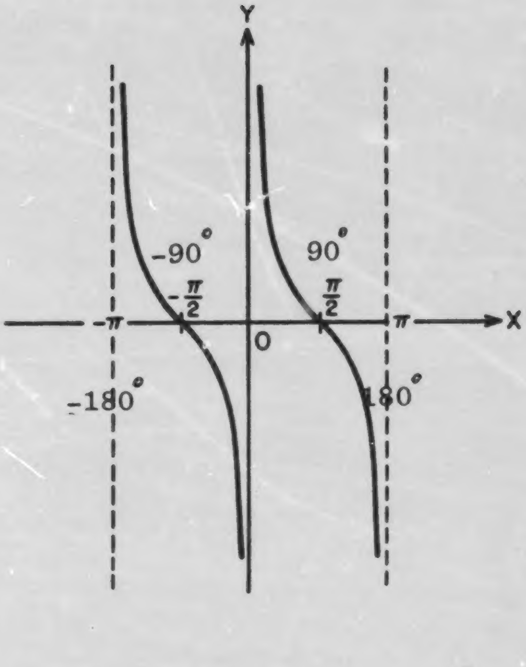
y = sin x



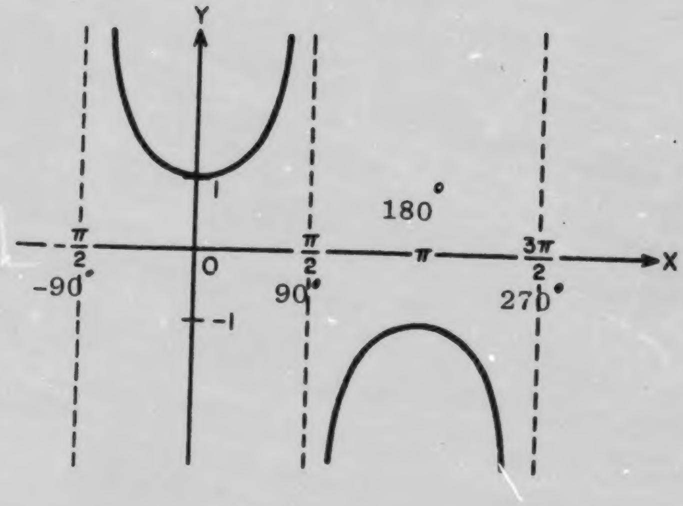
y = cos x



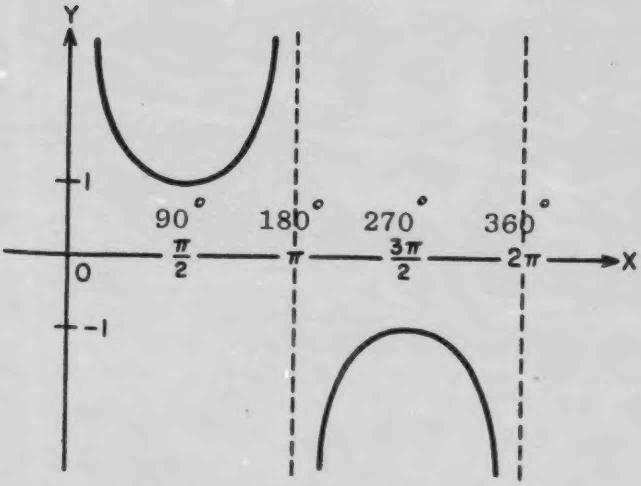
y = tan x



y = cot x


Y
1
180°
$\frac{\pi}{2}$
$\frac{\pi}{2}$
π
$\frac{3\pi}{2}$
X
-90°
0
90°
270°
-1

y = sec x


Y
1
90°
180°
270°
360°
0
$\frac{\pi}{2}$
π
$\frac{3\pi}{2}$
2π
X
-1

y = csc x

1) Graph:

a) $y = \cos 2x$

b) $y = \sin 2x$

c) $y = 2 \cos x$

d) $y = \sin \frac{1}{2}x + 2$

e) $y = \csc 2x$

2) What frequency results from a wave with $V = 10$ m/sec and $\lambda = 3$ m?

Trigonometric Curves - Applications

The previous section explained how and why a trigonometric curve is generated. This simple explanation of curves provided a sufficient introduction to the concept of "wavelengths" existing in nature.

Sound

Waves that carry "sound" must originate from a source, and be transmitted via some type of elastic medium. For instance, if a tuning fork is activated, a sound can be heard. But if the tuning fork is activated within a vacuum (where air does not exist), there will be no sound.

When a guitar string is plucked, its vibration of the string illustrates the sound wave. The string "oscillates". From this, the term "frequency" is used, defined as the number of times an oscillation will occur, per unit of time. Usually, cycles per second, denote the unit of frequency.

Sound waves travelling in air have the same frequency as sound waves travelling through a solid, like the guitar string.

The formula representing frequency of a sound wave can be expressed as:

$$f = \frac{\text{speed of the wave}}{\text{wavelength, or period}} = \frac{v}{\lambda}$$

By the technique of conversion factors, the formula can be manipulated as also represented as:

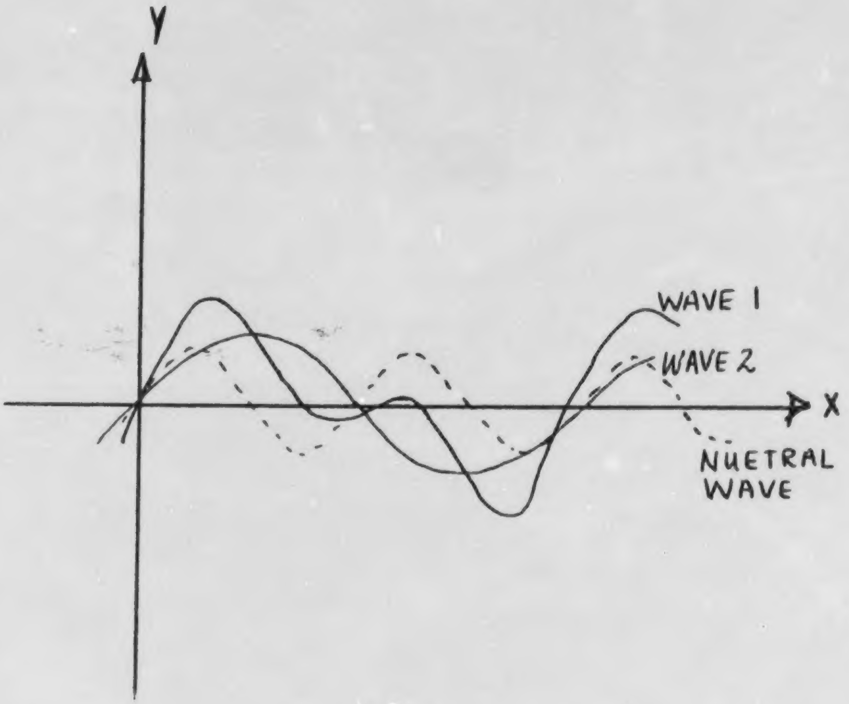
$$\lambda f = v \quad \text{or} \quad \lambda = \frac{v}{f}$$

The simplest type of a continuous, regular wave, is the *basic sine curve*, also known as a sinusoidal wave motion.

If we consider a guitar string, attached at both ends, plucking will create a vibration. Each time a period, or wavelength, is completed during the vibration, the wave is said to have moved a distance in the x direction. The wave speed, represented by v, can be expressed:

$$v = \frac{\text{wavelength, or period}}{\text{time, or period of oscillation}} = \frac{\lambda}{T}$$

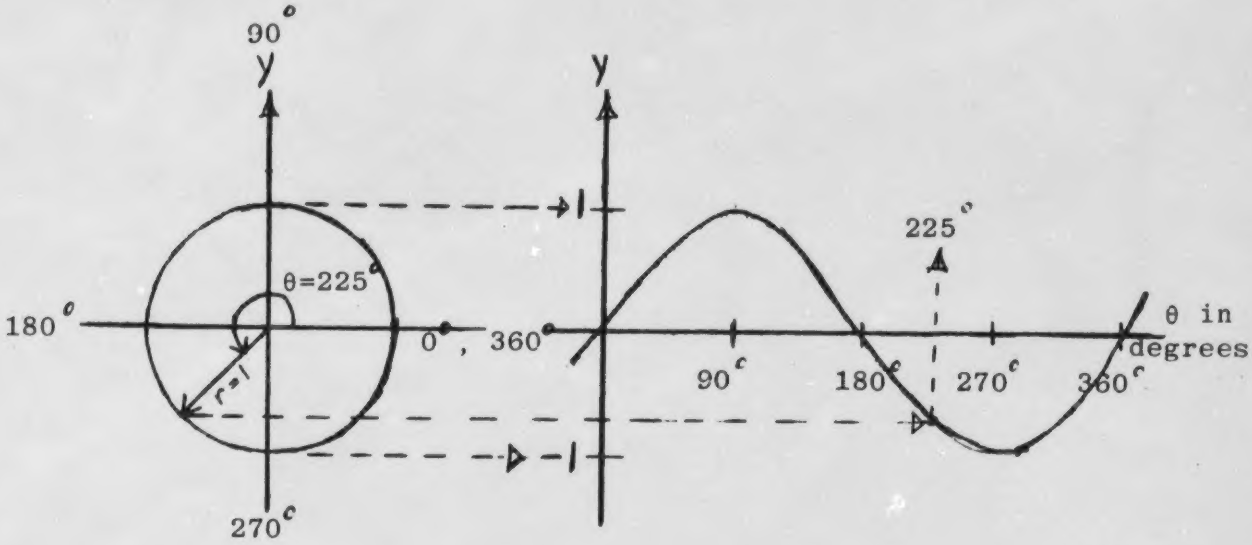
Two or more sine waves can travel simultaneously. If we consider two such waves, at the same frequency and amplitude (vertical distance from x axis to wave peak), each wave oscillates as if the other wave did not exist. However, a phenomenon occurs, which is known as interference. It is referred to as a phenomenon because even though each wave is acting independently, without knowledge of the other, the two waves begin to "neutralize" when they meet (as represented by the drawing). This "neutralizing" produces a combination of dissimilar pulses (waves).



The Sine Wave - Relationship to Vectors

The basic sine wave was found to peak at 90°, y = 1, and then form a trough at 270°, where y = -1.

A vector, revolving about an origin on a set of axes, x and y, can produce a basic sine wave. For instance, by comparing the two illustrations below, we have:



When the vector (on the left side of diagram), rotates to a value $\theta = 225°$, it corresponds to the value on the sine curve (on the right) at $\theta = 225°$. This will occur only when the vector is the same vertical distance from origin to circular path, as the amplitude distance of the sine wave.

Alternating currents and voltages produce a wave that is sinusoidal (basic sine wave). Also, from the study of voltages, we can relate to the "resultant" vector.

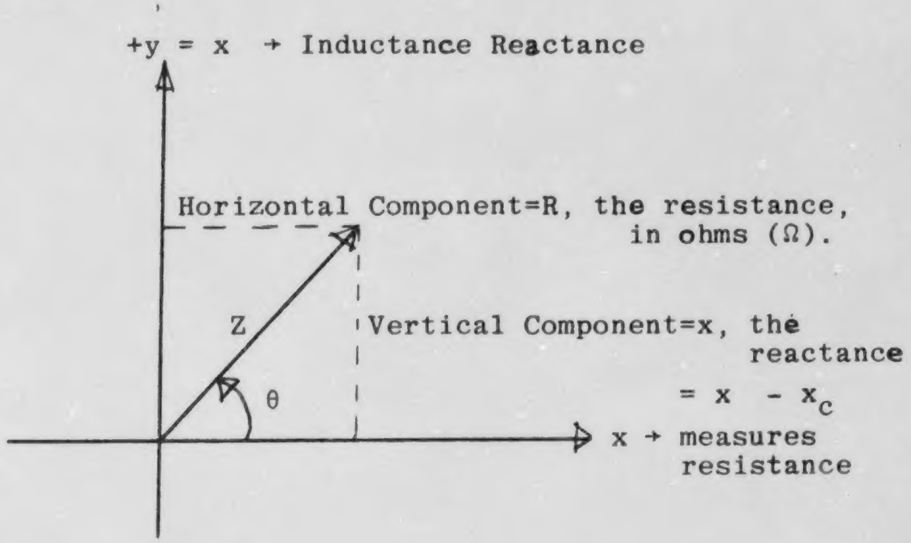
It was shown in the explanation of vectors, that resultants are formed by "tip to tail method" addition, or from calculation of horizontal and vertical components.

In an alternating current series circuit, impedance can be represented as a resultant vector. By definition, impedance is the square root of the "resistance squared, plus the difference, squared, between inductive reactance and capacitive reactance". Symbolically:

$$\text{Impedance} \rightarrow Z = R^2 + (x - x_c)^2$$

Note, R^2 in the equation is not "resultant squared". It is resistance, in ohms.

To denote impedance as a resultant vector, the set of axes x and y take on new meaning:


+y = x → Inductance Reactance
Horizontal Component=R, the resistance, in ohms (Ω).
Z
θ
Vertical Component=x, the reactance
= x - x_c
x → measures resistance

CALCULUS

Chapter V - Calculus

1. An Introduction
    - Differentiation
    - Integration
2. Differential Calculus
3. The Derivative - Applications
4. Integral Calculus
5. Area of Right Triangles Using Integration

CALCULUS

An Introduction

Calculus is the branch of mathematics involving two concepts of study:

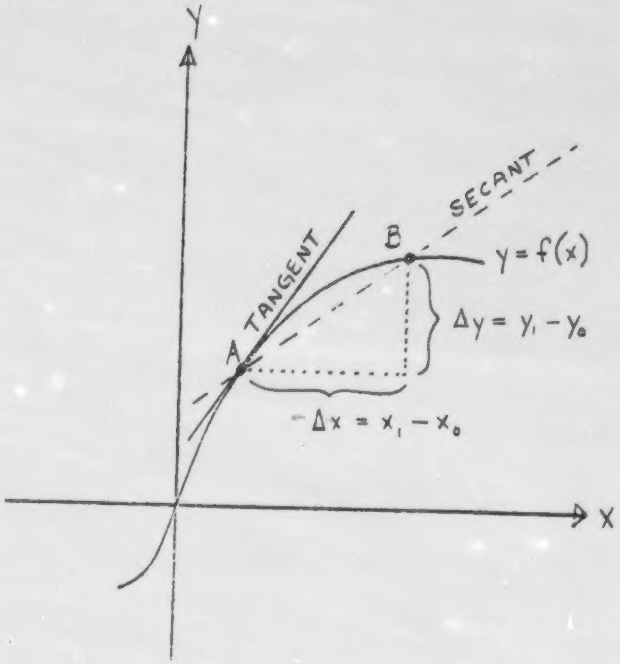
I. Differentiation
II. Integration

The procedure of learning differential calculus is usually introduced first. However, from a historical view, the process of differentiation was created after the art of integral calculus was developed. And, it should be noted that it was many years later that the two concepts were observed to be related to each other.

Differential calculus is sometimes referred to as "the change" of constantly varying quantities. These quantities could be instantaneous velocity, work acting on an object, or the maximum value of profit/loss in a business, as well as the minimum value of cost.

The derivative of a function is directly related to the slope of a tangent line at a point on a curve.

EXAMPLE:


Y
X
TANGENT
SECANT
A
B
$y = f(x)$
$\Delta y = y_1 - y_0$
$\Delta x = x_1 - x_0$

Note: x_0 and y_0 represent the initial point coordinates of A.

In the diagram above, we have $y = f(x)$, to represent any curve. Choosing a point A on the curve, we then construct a tangent line at point A.

At another place on the curve, we label point B. If a broken line is drawn connecting points A and B, it is called a secant line.

If the slope of the secant line is found, the slope of the tangent line at point A will be the first derivative of the secant line as Δx approaches zero.

As the "change in x", Δx, approaches zero, we imagine point B 'sliding' down the curve to point A. When the two points coincide, a state of tangency exists. Therefore, the slope of the secant line, $\frac{\Delta y}{\Delta x} \rightarrow \frac{y_1 - y_0}{x_1 - x_0}$, approaches the value of the tangent line at point A, as Δx approaches zero. It should be stated that the smaller Δx, or rather the closer together of points A and B, a better slope approximation will result.

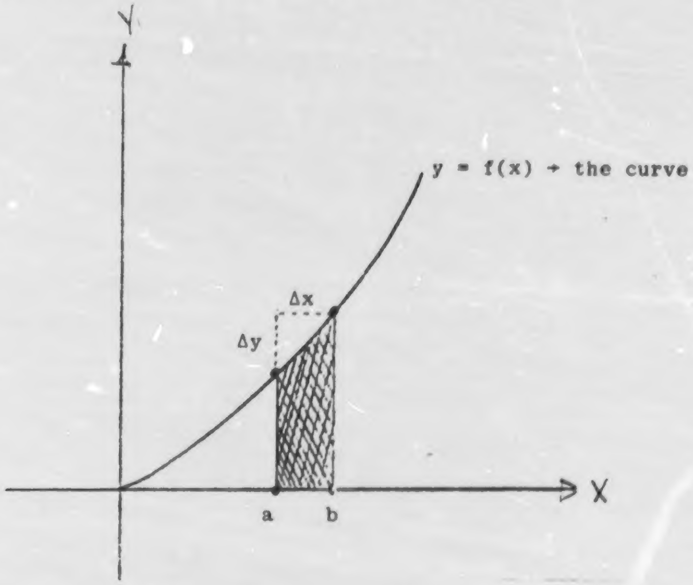
Basically, the ratio of $\frac{\Delta y}{\Delta x}$ between the change in x, Δx, and the change in y, Δy, from point A to B on any curve, will provide the slope of the tangent line, the first derivative. It is usually referred to as "the slope of the curve at point A". Notation for the derivative is:

If $y = f(x)$, then the first derivative of y with respect to x is expressed:

$$\frac{dy}{dx} = f'(x) = D_x y$$

As mentioned earlier, integral calculus was the first calculus to be discovered. After years of observation, integration was found to be the inverse operation of differentiation. Because of this reason, the "integral" is also referred to as the "antiderivative". Also, the converse of our statement is true: The inverse of differentiation is integration.

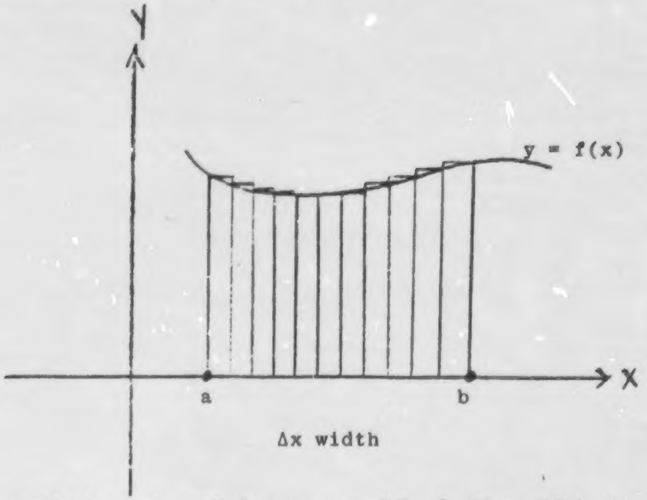
Integral calculus is the study relating to areas under, or bounded by a curve (s).



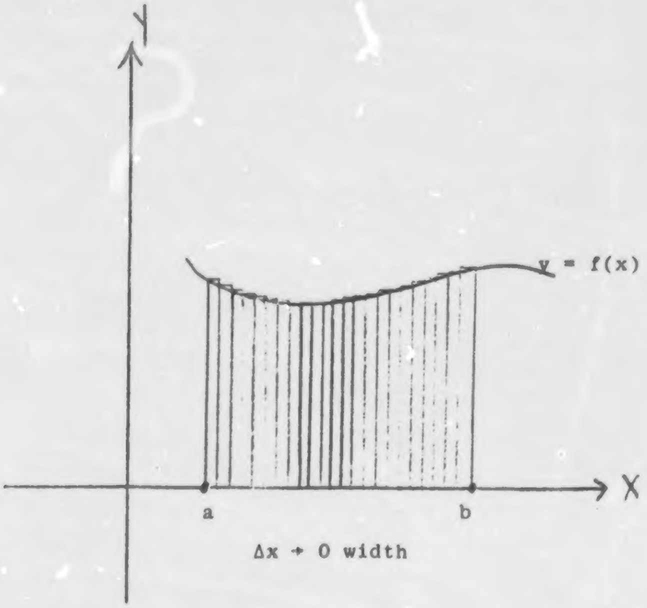
For example, the diagram shows a curve, y = f(x). Basically, integral calculus is utilized to find the area under the curve, in this case, from point a to b.

360

Consider the diagram:


Y
y = f(x)
X
a
b
Δx width

The numbers a and b are called the "limits of integration". The region bounded by the curve and points a and b, is divided into n rectangles, or partitions. If the region were divided into smaller n rectangles, with a smaller width of Δx, the diagram would be:


Y
y = f(x)
X
a
b
Δx → 0 width

The smaller the rectangles, the closer together they become, but the number of rectangles increase. In mathematical terms, as the number of rectangles, n, within a region gets large, the width of each rectangle, Δx, approaches zero. Therefore, the sum of rectangles tends to a limit as "n" gets large and each Δx approaches zero. Roughly speaking, this limit is called the definite integral of the curve $y = f(x)$ over the interval, a, b . In notation:

$$\int_a^b f(x)dx$$

Read, "the integral from a to b of f(x)dx". "dx" comes from "differentiation" and "delta x" → Δx, meaning the "change in x"; it is always part of each integration step.

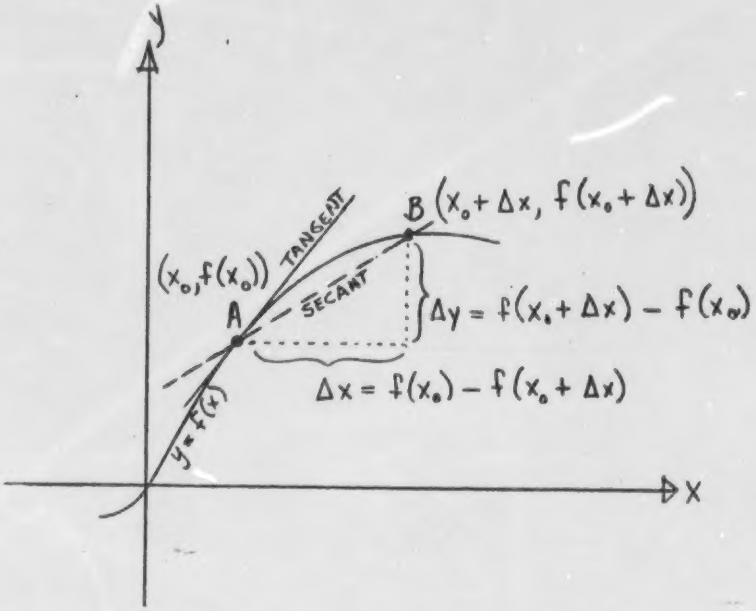
So, the concept of integration is actually the "pulling together" of partitions enclosed by a curve, in order to find the area.

Vertical rectangles only do not have to be used. Any geometric figures could possibly work.

The illustrations and explanations provided in this section were introduced in the most simplistic manner. Important theory was ommitted. However, the intent of the section was to provide a general definition of integral and differential calculus. For more complete explanations, the student should consult with an instructor and also refer to any calculus text.

Differential Calculus

The introduction explained the geometric interpretation of the derivative. If we use the same graph, add more details, we can see how the derivative results.


y
x
TANGENT
SECANT
$B\ (x_0+\Delta x,\ f(x_0+\Delta x))$
$(x_0, f(x_0))$
A
$\Delta y = f(x_0+\Delta x) - f(x_0)$
$\Delta x = f(x_0) - f(x_0+\Delta x)$
$y = f(x)$

We have described the coordinates of each point and the definition of Δx and Δy in terms of functions. The notation x_0 only means the "initial value of x before a change in x, Δx takes place.

Explaining the labels on the graph:

Point A → $(x_0, f(x_0))$

The first coordinate of Pt.A is the initial value of x. The second coordinate of Pt.A is the value of the function (curve) at x.

Point B → $(x_0 + \Delta x,\ f(x_0 + \Delta x))$

The first coordinate of Pt.B is the initial value of x <ins>plus</ins> the Δx from A to B. The second coordinate of Pt.B is the value of the function (curve) at $(x_0 + \Delta x)$.

$\Delta x \rightarrow\ f(x_0) - f(x_0 + \Delta x)$

$\Delta y \rightarrow\ f(x_0 + \Delta x) - f(x_0)$

From these coordinates and Δx with Δy:

$$\text{SLOPE OF SECANT LINE} \rightarrow M = \frac{\Delta y}{\Delta x} = \frac{y_1 - y_0}{x_1 - x_0} = \frac{f(x_0 + \Delta x) - f(x_0)}{(x_0 + \Delta x) - x_0}$$

Referring to the slope of the secant line, as Δx approaches zero, the limit of $\frac{\Delta y}{\Delta x}$ will yield the first derivative:

$$f'(x\ ) \rightarrow \begin{array}{l}\text{SLOPE OF TANGENT}\\ \text{LINE (SLOPE OF}\\ \text{THE CURVE)}\end{array} = \lim_{\Delta x \to 0} \frac{f(x_0 + \Delta x) - f\ (x_0)}{\Delta x}$$

Example:

Calculate the derivative at the indicated value of x.

$$\boxed{f'(x) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}}$$

Given: $f(x) = -2x + 3$, at $x = 1$

Solution:

We already know $f(x)$. In order to use the formula for derivative, we need $f(x + \Delta x)$:

$$f(x) = -2x + 3$$

$$f(x + \Delta x) = -2(x + \Delta x) + 3$$

Therefore:

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{\{-2(x + \Delta x) + 3\} - (-2x + 3)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{\{-2x - 2\Delta x + 3\} - (-2x + 3)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{-2x - 2\Delta x + 3 + 2x - 3}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{-2\Delta x}{\Delta x}$$

$$= -2$$

The derivative, or slope of the tangent line at $x = 1$, is -2.

By observation, the slope of the line $f(x) = -2x + 3$, is -2. We know this because slope-intercept formula, $y = mx + b$, states the slope is the coefficient of x. But, by the definition of derivative, we also found the value of the slope.

Since we can find the slope of the curve at an indicated x value, we can also determine the equation of the tangent line utilizing the first derivative.

From linear equations, we know:

$$y - y_1 = m(x - x_1)$$

The "point-slope" formula for finding the equation of a line. With this equation in mind, we have the following:

$$y - f(x_1) = f'(x_1)(x - x_1)$$

(EQUATION OF THE TANGENT LINE)

EXAMPLE:

Calculate the equation of the tangent line at the indicated point.

Given: $f(x) = x^2 + 1$ at $(1, 2)$

Solution:

First, if $f(x) = x^2 + 1$, then

$$f(x + \Delta x) = (x + \Delta x)^2 + 1$$

$$= x^2 + 2x\Delta x + (\Delta x)^2 + 1$$

then,

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{(x^2 + 2x\Delta x + \Delta x^2 + 1) - (x^2 + 1)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{x^2 + 2x\Delta x + \Delta x^2 + 1 - x^2 - 1}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{2x\Delta x + \Delta x^2}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{\Delta x(2x + \Delta x)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} 2x + \Delta x$$

$$= 2x$$

$= 2$ at $x = 1$ from pt. $(1, 2)$

Therefore, the equation of the tangent line at point (1, 2) with $f'(x) = 2$ is:

$$\begin{array}{rcll} y - f(x_1) & = & f'(x)(x - x_1) & \\ y - 2 & = & 2(x - 1) & \text{By substitution} \\ y - 2 & = & 2x - 2 & \text{Distributive Property} \\ +2 & & +2 & \\ \hline y & = & 2x & \end{array}$$

Finding Slope of a Curve at a Given Point

Using derivatives, we can easily find the slope of a given curve at a given point. For example:

$$y = 3x^2 - 2x \quad \text{at} \quad (5, 8)$$

$$\frac{dy}{dx} = \frac{d(3x^2)}{dx} - \frac{d(2x)}{dx}$$

$$\frac{dy}{dx} = 6x - 2$$

Therefore,

$$\text{Slope} \rightarrow \left.\frac{dy}{dx}\right| \text{at } x = 5 \text{ is } 6(5) - 2$$

$$30 - 2$$

$$28$$

Using derivatives, find the slope of the curve at the given point:

1) $y = x - x^{\frac{1}{2}}$ at $(4, 2)$

2) $y = \ln x$ at $(2, 0)$

3) $y = 3x^2 - 2x$ at $(\frac{1}{4}, 3)$

4) $y = \log x$ at $x = 5$

5) $y = 4^x$ at $x = -1$

Derivative Notation

The differentiation of functions can be done in a shorthand method. We begin by stating the formula for the derivative of an x^n function:

Given a function, $y = x^n$:

$$f'(x) = \frac{d(x^n)}{dx} = nx^{n-1}$$

For example, we take the first derivative of the listed functions:

1) If $f(x) = x^3$, then $f'(x) = (3)x^{3-1} = 3x^2$

2) If $f(x) = x^{42}$, then $f'(x) = (42)x^{42-1} = 42x^{41}$

3) If $f(x) = x^1$, then $f'(x) = (1)x^{1-1} = x^0 = 1$

4) If $f(x) = x^{-5}$, then $f'(x) = (-5)x^{-5-1} = -5x^{-6}$

Next, suppose we have a function that is multiplied by a constant, c:

Given a function, $f(x) = y = cx^n$:

$$f'(x) = \frac{dy}{dx} = (n)(c)x^{n-1}$$

Examples:

1) If $f(x) = 3x^2$, then $f'(x) = (3)(2)x^{2-1} = 6x^1$

2) If $f(x) = \frac{1}{3}x^4$, then $f'(x) = (\frac{1}{3})(4)x^{4-1} = \frac{4}{3}x^3$

3) If $f(x) = 2x^{-\frac{1}{4}}$, then $f'(x) = (2)(-\frac{1}{4}x^{-\frac{1}{4}-1} = -\frac{1}{2}x^{-\frac{5}{4}}$

4) If $f(x) = 2x^{-3}$, then $f'(x) = (2)(-3)x^{-3-1} = -6x^{-4}$

Derivatives - Summary of Rules

I. If $f(x) = x^n$, then $f'(x) = (n)x^{n-1}$

II. If $f(x) = cx^n$, then $f'(x) = (n)(c)x^{n-1}$
(Also known as the power rule)

III. If $f(x) =$ constant, then $f'(x) = 0$

IV. Derivatives can be taken, term by term:

$$\frac{d}{dx}(f(x) \pm g(x)) = \frac{df(x)}{dx} \pm \frac{dg(x)}{dx}$$

The following rules applied to more advanced problems in differentiation.

I. $\frac{d(\ln x)}{dx} = \frac{1}{x}$

II. $\frac{d(e^x)}{dx} = e^x$

III. $\frac{d(\log x)}{dx} = \frac{1}{x}(0.4343)$

IV. $\frac{d(a^x)}{dx} = a^x(\ln a)$

Using differentiation, find $\frac{dy}{dx}$, or f'(x) of the following:

1) $y = \frac{1}{2}x^2$

2) $y = 3x^2 - 2x$

3) $y = 5x^{-4}$

4) $y = 4$

5) $y = x^{\frac{3}{2}}$

6) $f(x) = 2x^2 + 5x - 6$

7) $f(x) = -x^2$

8) $f(x) = \frac{2}{3}x$

9) $f(x) = x^3 - 7$

10) $f(x) = - 5x$

After finding the first derivative, evaluate each derivative from $x = 1$ to $x = 4$.

The Derivative - Applications

Common applications of the derivative include determining minimum surface area, minimum cost of production, minimum amount of building materials necessary, marginal cost, and marginal average cost.

For instance, from linear equations, we have the Total Cost function:

$$C = C(x) = mx + F$$

Marginal cost is represented by the slope, m, of the linear function. The definition of derivative is $\frac{\Delta y}{\Delta x}$, which also describes the slope. Therefore, the first derivative of C, denoted $\frac{dC}{dx}$, or C', is:

$$C'(x) = m$$

Marginal revenue, the change in total revenue when one more unit is produced, is the first derivative of R, the revenue, denoted:

$$R'(x)$$

By using the derivative, we not only determine the marginal cost and marginal revenue, but also the conditions required for maximum profit. For example:

If a firm has a Total Cost operation represented by $C = 0.01x^2 + 0.6x + 20$, and $R = 4x$, determine the following by use of derivatives.

Marginal Cost → $\frac{dC}{dx} = \frac{d(0.01x^2)}{dx} + \frac{d(0.6x)}{dx} + \frac{d(20)}{dx}$

$$\frac{dC}{dx} = 2(0.01)x^{2-1} + 1(0.6x^{1-1}) + 0$$

$$\frac{dC}{dx} = 0.02x + 0.6x^0$$

$$\frac{dC}{dx} = 0.02x + 0.6$$

Marginal Revenue → $\frac{d(R)}{dx} = \frac{d(4x)}{dx}$

$$\frac{d(R)}{dx} = 4$$

To find the conditions for maximum profit, set $\frac{d(C)}{dx} = \frac{d(R)}{dx}$:

$$0.02x + 0.06 = 4$$

Solve for x:

$$\begin{array}{rcl} 0.02x + 0.06 & = & 4 \\ -\ 0.06 & & -\ 0.06 \\ \hline 0.02x & = & 3.94 \end{array} \qquad x = 197$$

By solving for $x = 197$, this means that to achieve a maximum profit, the production should not exceed 197 units of a particular item.

Another interesting problem involves determining the minimum amount of fence to enclose a given area.

Suppose we have a rectangular garden plot, consisting of an area of 10,000 sq. ft. To find the minimum amount of fence required to enclose the area on all four sides, we need to know the following:

$$\text{Area} = (x)(y) \qquad \text{Fencing Required} = 2x + 2y$$

Express y in terms of x before differentiating.

$$A = (x)(y)$$

$$10{,}000 = (x)(y)$$

$$\frac{10{,}000}{x} = y$$

Substitute the value for y into "Fencing Required":

$$\text{Fencing Required} = 2x + 2\left(\frac{10{,}000}{x}\right)$$

$$\text{F. R.} = 2x + \frac{20{,}000}{x}$$

$$\text{F. R.} = 2x + 20{,}000x^{-1}$$

In order to find the minimum amount of fence required, first take the derivative of F.R., with respect to x:

$$\frac{d(F.R.)}{dx} = \frac{d(2x)}{dx} + \frac{d(20,000x^{-1})}{dx}$$

$$\frac{d(F.R.)}{dx} = 2 + (-1)(20,000)x^{-1-1}$$

$$\frac{d(F.R.)}{dx} = 2 - 20,000x^{-2}$$

or

$$\frac{d(F.R.)}{dx} = 2 - \frac{20,000}{x^2}$$

Set $\frac{d(F.R.)}{dx}$ equal to zero, and solve for x:

$$\frac{d(F.R.)}{dx} = 0$$

$$2 - \frac{20,000}{x^2} = 0$$

$$2x^2 - 20,000 = 0$$

$$2x^2 = 20,000$$

$$x^2 = 10,000$$

$$x = \sqrt{10,000} = 100$$

The minimum amount of fencing required is obtained after substituting the x value into F.R.

$$\text{Minimum Fencing Frquired} = 2x + 2\left(\frac{10,000}{x}\right)$$

$$= 2(100) + 2\left(\frac{10,000}{100}\right)$$

$$= 200 + 200$$

$$= 400 \text{ ft.}$$

EXAMPLE: If a manufacturer can produce 80 cubic inches of soup in a cost effective manner, what is the minimum size a can should be to hold 80 cubic inches of product?

SOLUTION: This type of problem also involves the use of derivatives.

We are first concerned with "**surface area**" of a cylindrical container. Surface area refers to the amount of material included on top, bottom, and the curved portion.

By formula definition:

$$\boxed{\text{Surface Area} \rightarrow S = 2\pi r^2 + 2\pi rh}$$

We also need to know the formula of volume pertaining to a cylindrical can:

$$\boxed{\text{Volume} \rightarrow V = \pi r^2 h}$$

In our problem, the container must hold 80 cubic inches; this is the volume. Therefore,

$$V = \pi r^2 h$$

$$80 = \pi r^2 h$$

Volume can also be expressed as:

$$\frac{80}{\pi r^2} = h$$

If we substitute this value for h, into the equation of "surface area", we have:

$$\text{Surface Area} \rightarrow S = 2\pi r^2 + 2\pi r h$$

$$S = 2\pi r^2 + 2\pi r\left(\frac{80}{\pi r^2}\right)$$

$$S = 2\pi r^2 + \frac{160\pi r}{\pi r^2}$$

$$S = 2\pi r^2 + 160 r^{-1}$$

To find the minimum surface area of the can, take the first derivative, with respect to r:

$$S = 2\pi r^2 + 160 r^{-1}$$

$$\frac{d(S)}{dr} = \frac{d(2\pi r^2)}{dr} + \frac{d(160 r^{-1})}{dr}$$

$$\frac{d(S)}{dr} = 4\pi r - 160\ r^{-2}$$

Set $\frac{d(S)}{dr}$ equal to zero, and solve for r:

$$\frac{d(S)}{dr} = 0$$

$$4\pi r - 160r^{-2} = 0$$

$$4\pi r - \frac{160}{r^2} = 0$$

$$4\pi r^3 - 160 = 0$$

$$4\pi r^3 = 160$$

$$r^3 = \frac{160}{4\pi}$$

$$r^3 = \frac{160}{12.566}$$

$$r^3 = 12.73277$$

$$r \doteq 2.35 \text{ inches}$$

Therefore, the minimum dimensions of a can to hold 80 cubic inches of soup would be:

$$\text{radius} \doteq 2.35 \text{ inches}$$

$$\begin{aligned}\text{height} &\doteq \frac{80}{\pi r^2}\\ &\doteq \frac{80}{\pi(2.35)^2}\\ &\doteq \frac{80}{17.35}\\ &\doteq 4.61 \text{ inches}\end{aligned}$$

Using differentiation, evaluate the following:

1) A firm has a Total Cost function of:

$$C = 0.1x^{\frac{1}{2}} + 3x + 8$$

with revenue, $R = 5x$.

Determine: (a) marginal cost
(b) marginal revenue
(c) conditions for maximum profit

2) If a rectangular garden plot has an area of 5,125 sq. ft., what is the minimum amount of fencing required to enclose the area on all four sides?

3) Referring to the garden plot in problem #2, what is the minimum amount of fencing required if we want to place a fence through the middle, as well as on all four sides?

4) A manufacturer can produce 165 cubic inches of product in a cost effective manner. What is the minimum size that a can should be in order to hold 165 cubuc inches of product?

5) A firm has a cost function:

$$C = 0.01x^2 + 0.6x + 20$$

and revenue, $R = 7x$

Determine: (a) marginal cost
(b) marginal revenue
(c) conditions for maximum profit

Integral Calculus

The Fundamental Theorem of Calculus is the calculus first discovered, integration. The theorem states that an area bounded by a curve between limits a and b can be found by the antiderivative, or definite integral.

There has been controversy through the years concerning who developed the first Calculus, later known as integration. Scholars will testify that Gottfried Leibniz (b. 1646 - Leipzig, Germany) invented the Calculus, while others state Issac Newton (b. 1642 - Woolsthorpe, England) invented the procedure of finding an area bounded by a curve. The controversy still exists, however many have resolved to say Newton and Leibniz invented the Calculus independently. Many believe "fuel to the fire" began when the work of Leibniz was published before Newton.

Integration is the inverse operation of differentiation. Because of this, the integral is sometimes referred to as the "antiderivative".

Explaining by example, if we consider a function,

$$f(x) = 3x^4 + 5$$

the derivative of f(x) is:

$$f'(x) = 12x^3$$

Taking the antiderivative, or integral of f'(x):

$$\text{Integral of } 12x^3 = 3x^4 + 5$$

By correct notation, we have:

$$12x^3dx = 3x^4 + 5$$

Notice there are no limits, a and b, on the integral symbol as stated in the definition. We call an integral, indefinite, if boundaries are not specified.

Because the integral and the derivative are inverse operations of each other, we will state both definitions for the student to observe and compare:

Given $y = f(x) = x^n$, then the definitions of:

INTEGRATION → $\int x \, dx = \frac{x^{n+1}}{n+1} + C, \; n \neq -1$

DIFFERENTIATION → $\frac{d\left(\frac{x^{n+1}}{n+1}\right) + C}{dx} = x^n, \; n \neq -1$

It is important to note that each time the indefinite integral of a function is taken, the result (answer) will always include "+C". This arbitrary constant, C, is part of the answer by convention. It is referred to as the constant of integration. When taking a definite integral (with specified boundaries a and b), the fundamental theorem of Calculus permits us to select any value for C, usually C = 0 is chosen. Therefore, the "+C" does not appear in the answer.

Explanation of Difference Between Definite and Indefinite Integral

Given a function, f(x), the indefinite integral would be represented as $\int f(x)dx$, where limits a and b do not exist. Further explanation states that the integral of f(x) results in another function, having a derivative equal to f(x). Symbolically,

$$\boxed{\int f(x)dx = g(x), \text{ where } \frac{dg(x)}{dx} = f(x)}$$

The definite integral of f(x), denoted $\int_a^b f(x)dx$, results in a value dependent upon the boundaries a and b, as well as the function itself.

Properties of the Definite Integral

The following theorems are statements which have been either proved to be true, or truths which are conjectural -- meaning there is some evidence that the theorems are true. The stated theorems involve functions considered to be "integral" from limit a to limit b.

Given, a function f(x) having limits a and b, with k → any constant, then:

$$\int_a^b k \cdot f(x)dx = k \int_a^b f(x)dx$$

To specify the sum and difference of two integral functions:

Given, functions f(x) and g(x), with limits a and b, then:

$$\int_a^b \{f(x) \pm g(x)\}dx = \int_a^b f(x)dx \pm \int_a^b g(x)dx$$

Additional Rules of Integration

$$\boxed{\int x^{-1}\,dx = \int \frac{1}{x}\,dx = \ln x + C}$$

$$\boxed{\int (\ln x)\,dx = x\,(\ln x) - x + C}$$

$$\boxed{\int a^{x}\,dx = \frac{a^{x}}{\ln a} + C}$$

$$\boxed{\int e^{x}\,dx = e^{x} + C}$$

Evaluating Definite Integrals

The definite integral has limits a and b. After the integral of a function is derived (calculated), the evaluation of the result is dependent upon the specified limits. Review the following examples:

EXAMPLE I:

Evaluate: $\int_0^4 x^2 dx$

Solution:

$$\int_0^4 x^2 dx = \left.\frac{x^{2+1}}{2+1} + C\right]_0^4$$

$$= \left.\frac{x^3}{3}\right]_0^4$$

$$\text{At } x = 4; \quad \frac{x^3}{3} = \frac{(4)^3}{3} = \frac{64}{3}$$

$$\text{At } x = 0; \quad \frac{x^3}{3} = \frac{(0)^3}{3} = 0$$

$$\text{Evaluation} \rightarrow \left.\frac{x^3}{3}\right]_0^4 = \frac{64}{3} - 0 = \frac{64}{3}$$

In the preceding example, the integral of the function x^2 was calculated. Next, the result was evaluated at the upper limit, 4, and then at the lower limit 0. The difference between the two will provide the final evaluation (answer).

EXAMPLE 2:

Evaluate: $\int_3^5 3x\ dx$

Solution: $\int_3^5 3x\ dx = 3\int_3^5 x\ dx$

$$= 3 \cdot \frac{x^{1+1}}{1+1} + C \Big]_3^5$$

$$= 3 \cdot \frac{x^2}{2} \Big]_3^5$$

$$= \frac{3}{2}x^2 \Big]_3^5$$

At $x = 5$; $\frac{3}{2}x^2 = \frac{3}{2}(5)^2$

$$= \frac{3}{2}(25)$$

$$= \frac{75}{2}$$

At $x = 3$; $\frac{3}{2}x^2 = \frac{3}{2}(3)^2$

$$= \frac{3}{2}(9)$$

$$= \frac{27}{2}$$

$$\text{Evaluation} \rightarrow \left.\frac{3}{2}x^2 + C\right|_3^5 = \frac{75}{2} - \frac{27}{2}$$

$$= \frac{48}{2}$$

$$= 24$$

Remember, the constant of integration, C, was dropped because the defivation of definite integrals {Fundamental Theorem of Calculus} permits that C can equal zero.

Integrate the following indefinite integrals.

1) $\int \ln x \, dx$

2) $\int 2^x \, dx$

3) $\int (x^{-2} + 3x^2) \, dx$

4) $\int (3 + \frac{2}{x}) \, dx$

5) $\int (2x - 3x^2 + 4x^3) \, dx$

6) $\int (x - 1) \, dx$

7) $\int x^{\frac{1}{2}} \, dx$

8) $\int 2 (\ln x) \, dx$

9) $\int 3 (2^x) \, dx$

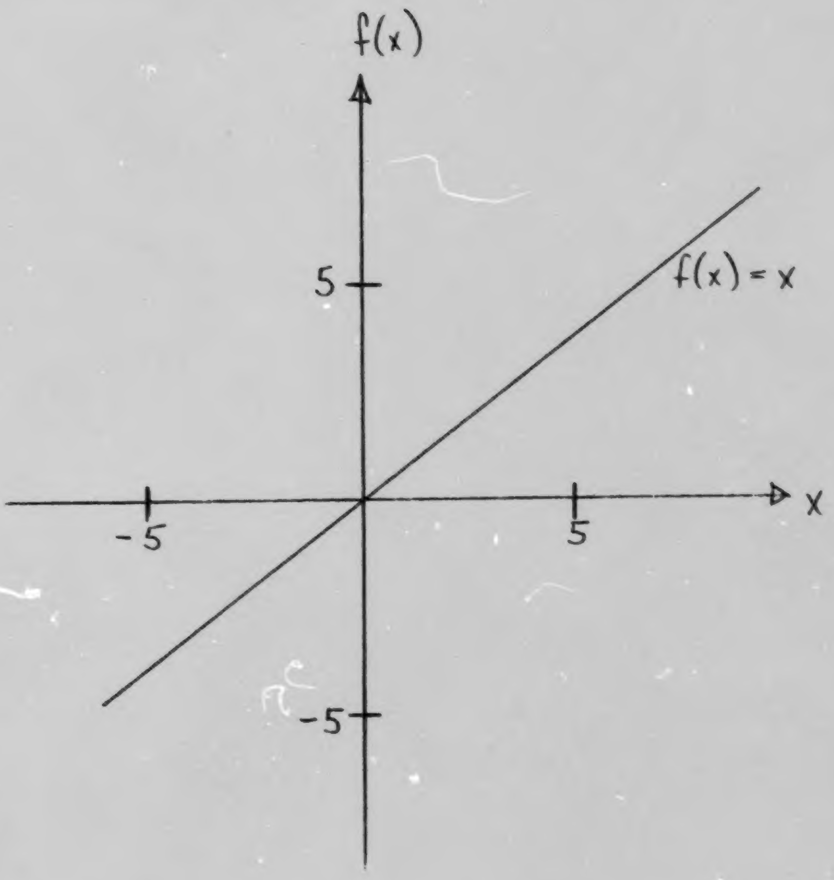
10) $\int 4 \ln x \, dx$

Now, evaluate the above integrals on a defined interval, from $x = 0$ to $x = z$.

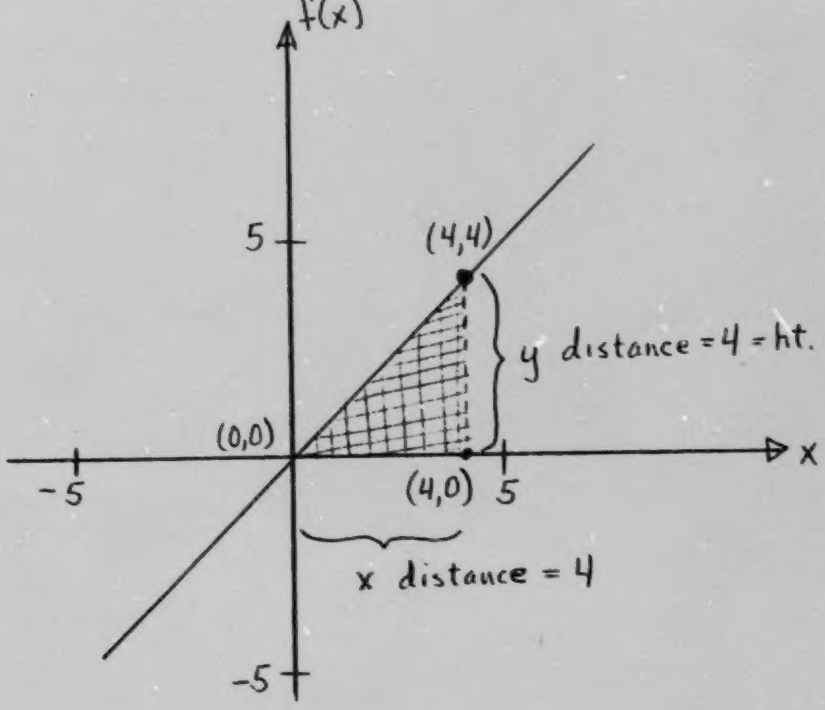
Area of Right Triangles Using Integration

Integral calculus is used to compute area bounded by curves using the process of integration. However, the technique can be applied to simple areas within right triangles.

If we consider a straight line, such as f(x) = x, passing through the origin, a graph would illustrate:



To form a right triangle, construct a vertical line from a point on the straight line, to a point on the x axis. For instance:


f(x)
5
-5
-5
5
x
(4,4)
(0,0)
(4,0)
y distance = 4 = ht.
x distance = 4

The shaded portion is a right triangle. Its area could be easily found by a process not utilizing integration:

$$\text{Area}_{\text{Rt. Triangle}} = \frac{1}{2}(\text{base})(\text{height})$$

$$A = \frac{1}{2}(4)(4)$$

$$A = \frac{1}{2}(16)$$

$$A = 8$$

With integration techniques, the area is computed:

$$\text{Area} = \int_0^4 f(x)\,dx$$

$$A = \int_0^4 x\,dx$$

$$A = \frac{x^2}{2}\Big]_0^4$$

$$A = \frac{4^2}{2} - \frac{0^2}{2}$$

$$A = \frac{16}{2} - 0$$

$$A = 8$$

Solving for Areas Bounded By Curves - Integration

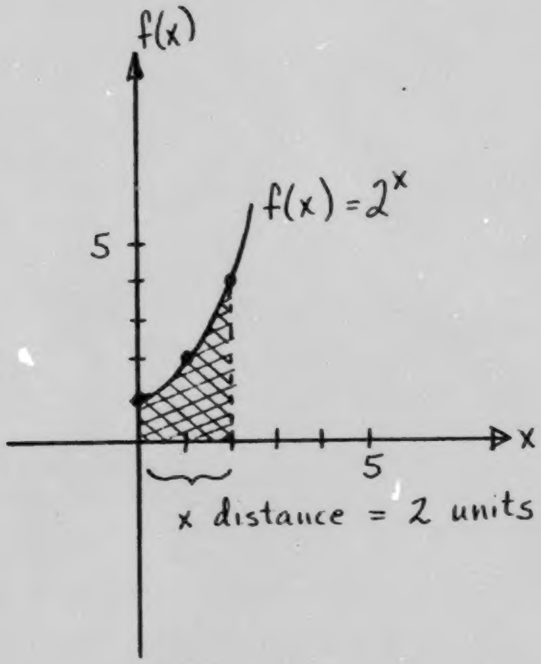
In this section, various curves will be introduced, with solutions provided for area bounded by the curves. Explanations concern approximating the areas within an interval on the positive x axis only. Intervals with negative values, and y axis values do exist. However, this type of discussion is beyond the scope of our presentation. Please refer to your instructor and/or any calculus text for a more detailed account of the integration process.

EXAMPLE:

Given the curve, $f(x) = 2^x$.

Draw the graph and approximate the area underneath the curve on an interval $x = 0$ to $x = 2$.

Solution:



$f(x) = 2^x$

$f(0) = 2^0 = 1$

$f(1) = 2^1 = 2$

$f(2) = 2^2 = 4$

$f(3) = 2^3 = 8$

$$\text{Area} = \int_{x=0}^{x=2} f(x)\,dx$$

$$\boxed{\int a^{x}\,dx = \frac{a^{x}}{\ln a} + c}$$

$$A = \int_{0}^{2} 2^{x}\,dx$$

$$A = \frac{2^{x}}{\ln 2}\Bigg]_{0}^{2}$$

$$A = \frac{2^{2}}{\ln 2} - \frac{2^{0}}{\ln 2}$$

$$A = \frac{4}{0.693} - \frac{1}{0.693}$$

$$A = \frac{3}{0.693} = 4.328 \text{ square units}$$

EXAMPLE:

Given the straight line, $f(x) = 6 - 2x$.

Draw the graph and approximate the area bounded by the line and the x and y axes.

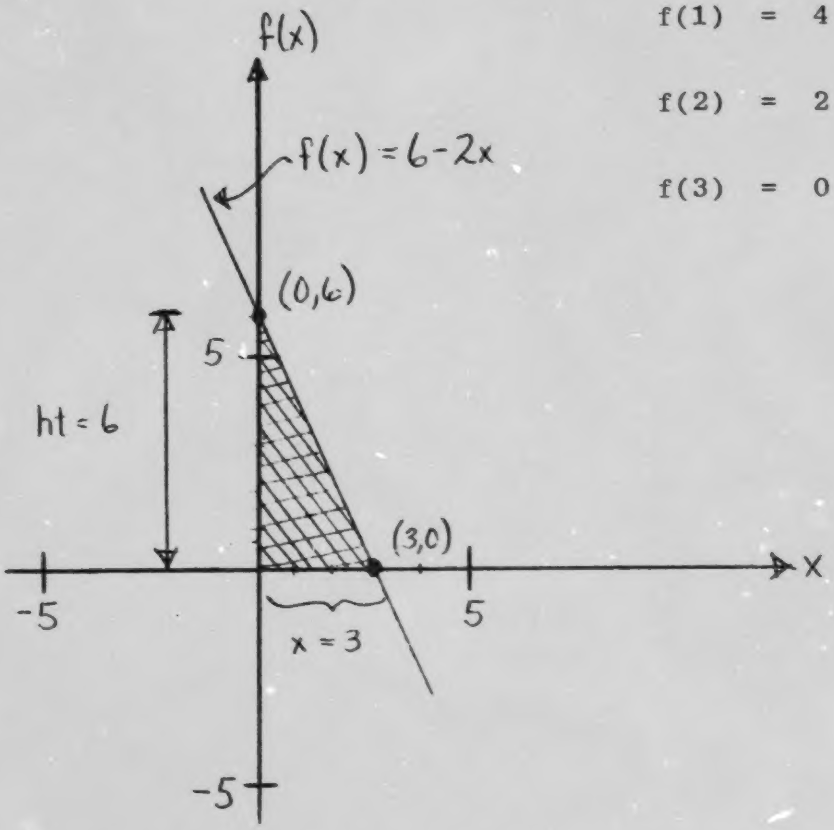
Solution:

$f(0) = 6$

$f(1) = 4$

$f(2) = 2$

$f(3) = 0$



The area forms a right triangle.

The interval on the x axis related to the area is $x = 0$ to $x = 3$.

$$\text{Area} = \int_{x=0}^{x=3} f(x)\,dx$$

$$A = \int_{0}^{3} 6 - 2x\,dx$$

$$A = 6\int_{0}^{3} dx - 2\int_{0}^{3} x\,dx$$

$$A = 6x - 2 \cdot \frac{x^2}{2}\Bigg]_{0}^{3}$$

$$A = 6x - x^2\Bigg]_{0}^{3}$$

$$A = \{6(3) - (3)^2\} - \{6(0) - (0)^2\}$$

$$A = \{18 - 9\} - \{0\}$$

$$A = 9 \text{ square units}$$

To check, calculate the area by conventional method:

$$\text{Area of Right Triangle} = \frac{1}{2}\,(\text{base})(\text{height})$$

$$A = \frac{1}{2}\,(3)(6)$$

$$A = \frac{1}{2}\,(18)$$

$$A = 9 \text{ square units}$$

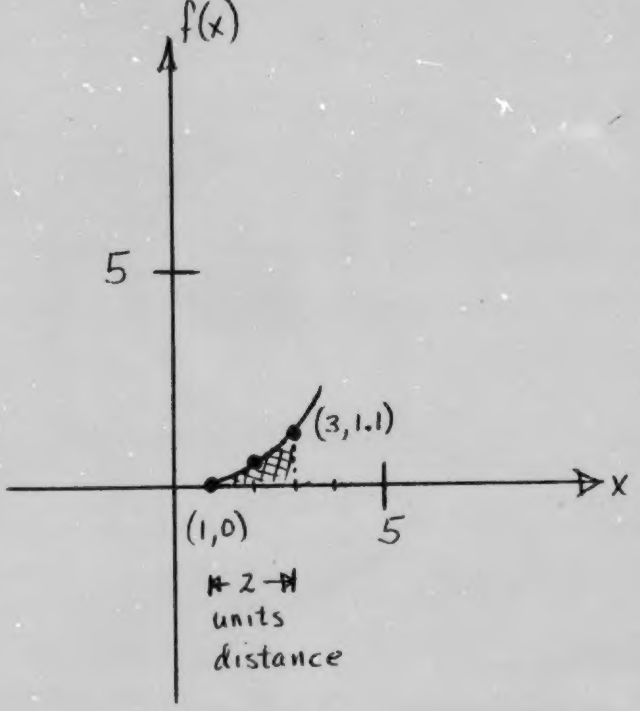
EXAMPLE:

Given the curve, $f(x) = \ln x$.

Graph the function.

Approximate the area underneath the curve on an interval from $x = 1$ to $x = 3$.

Solution:



$f(x) = \ln x$

$f(1) = \ln 1 = 0.0$

$f(2) = \ln 2 = 0.6$

$f(3) = \ln 3 = 1.1$

$$\text{Area} = \int_{x=1}^{x=3} f(x)\,dx$$

$$A = \int_{1}^{3} \ln x\,dx$$

$$\boxed{\int \ln x\,dx = x\,(\ln x) - x + c}$$

$$A = x(\ln x) - x \Big]_{1}^{3}$$

$$A = \{3(\ln 3) - 3\} - \{1(\ln 1)\ [illegible]\ 1\}$$

$$A = 0.2958 - (-1)$$

$$A = 0.2958 + 1$$

$$A = 1.2958 \text{ square units}$$

Using integration, find the plane areas which are bounded by the given equations:

1) $y = x^2, \quad y = 0, \quad x = 2, \quad x = 4$

2) $y = 5x - x^2, \quad y = 0, \quad x = 1, \quad x = 5$

3) $y = 16 - x^2, \quad y = x + 3$

4) $y = x^2 - 4, \quad y = 8 - 2x^2$

5) $y = 3x^2 - 9, \quad x = 0, \quad y = 0, \quad y = 1$

6) $y = x^3 - 6x^2 + 8x$ and the x axis

7) $y = 2x - 4, \quad y^2 = 4x$

8) $y = 6x - x^2$ and $y = x^2 - 2x$

9) $y = 2x^2$, x axis, $x = 1, \quad x = 3$

10) $y = 4x - x^2$ and the x axis

APPLICATIONS IN TECHNOLOGY

Chapter VI - Applications in Technology

The materials in this chapter were submitted to Project Director by various individuals. Each contribution was reviewed and edited.

Upon reading through each section, the importance of mathematics in technical fields can be easily seen. It is for this reason the sections were included.

1. Plane Surveying
2. Introduction to Computer Applications
   (Supplemented by Carol Flannery)
3. Machine Parts Inspection
4. Automotive Technology
5. Aviation - Basic Principles
6. Manufacture Engineering Processes
7. General Information for Solving Pharmacology Problems
8. Optical Technology

PLANE SURVEYING

Introduction

Surveying is the process of making angular and linear measurements and using the principles of geometry and trigonometry to define the location of points. Surveying is used to define property lines, insure alignment of construction, locate center-lines and right-of-ways for roads and railways, and to obtain contour maps, just to mention a few applications. In all of these uses, surveying involves three basic measurements: direction, distance and elevation.

Surveying is not strictly field work. It is not enough for a surveyer to be expert at manipulating surveying equipment. After field measurements are complete, the information must be presented in a convenient format. Often this involves calculation, corrections for systematic errors, and plotting.

We want to look at the basic concepts of plane surveying, and see the applications of mathematics in this important engineering area. We will consider three important types of surveying operations: distance measurement, determination of elevation, and traverse.

Distance Measurement

Measurement of lineal distance has been done since ancient times. One of the early methods of distance measurement that still has household application is pacing. In pacing operations, the length of an average step is measured. Then the number of steps along the line to be measured is counted and the distance is obtained by multiplying the size of each stride by the number of strides.

ex.

A garden plot is to be measured to find the approximate size, but only a yardstick is available. Tom measures the length of his average step and finds it to be 2.75'. He then counts the number of steps along the length of the garden and obtains the number 15.5 and along the width obtains 10. What are the approximate dimensions of the garden?

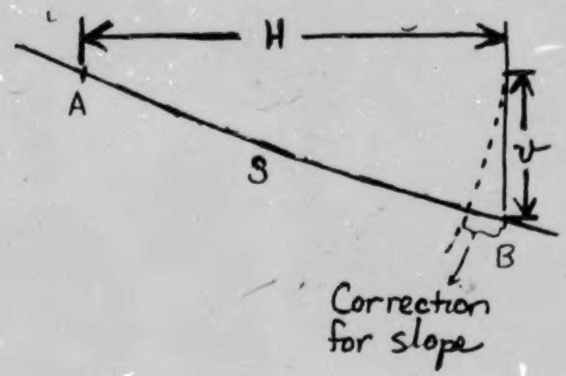
$$\text{Length} = 15.5 \text{ steps} \times \frac{2.75'}{\text{step}} = 42.6'$$

$$\text{Width} = 10 \text{ steps} \times \frac{2.75'}{\text{step}} = 27.5'$$

Pacing is a simple procedure and is commonly used for household applications. Applications in civil engineering are limited, however, because the method is inprecise. Over long distances, the step size tends to change, or the number of steps taken is forgotten. Other methods in distance measurement are usually used (which tend to reduce the total amount of error).

One method of distance measurement is taping. The word "chain" is usually used for a Gunter chain which is 66 feet long. The Gunter chain was developed because it facilitates measurement of acreage, a common land area measure. With a Gunter chain, 1 acre is 1 chain wide by 10 chains long. In other words, 1 acre is 66' x 660' or 43,560 square feet. Other types of chain have more general use than does the Gunter chain. The 100 foot and 200 foot tapes are most commonly used in civil engineering. Nickel-steel alloy tapes are available at somewhat higher cost and show a reduced tendency to elongate in hot weather. Woven tapes and fiberglass tapes are also available.

It is important to mention that, in general, we want to measure the horizontal distance between point A and B. If the ground surface is level, the tape can be laid out beginning at A. The tape lengths are marked until the distance to point B is covered. Often, however, measurement is taken on a sloping surface and the horizontal distance must be calculated. We want to obtain H, but we measure the slope distance S. The correction for slope is the difference between s and h. But s, h, and v are related according to the Pythagoreum Theorem $s^2 = h^2 + v^2$.



From algebra $s^2 - h^2 = v^2$,

and $(s + h)(s - h) = v^2$.

Then $s - h = \frac{v^2}{s+h}$

This is the correction that we want, but the form of the correction is not easy to apply because we would have to know h in order to solve $v^2/s+h$. We know however that s and h are approximately the same size, so we can use the expression $v^2/2s$ to obtain the horizontal distance when the slope distance is measured.

Because the slope distance, s, is larger than the horizontal distance, h, the correction is always subtracted when h is calculated.

Lets consider one more type of correction that is used in taping. When a tape is new, the manufacturer checks the length of the tape against a standard distance to assure that the tape is accurate. However, tapes can and do break in the field and must be repaired. The two ends are often overlapped and welded. This process can result in the tape not being the standard length.

ex.

A distance of 785.62' is measured using a 100' steel tape. The tape is later compared to a standard and is found to be 99.92' long. Find the corrected distance.

For this example, we can see that every time we used the tape we considered that 100.00' had been measured. Actually a shorter distance had been measured, so we can reason that the correction must be subtracted if the tape is too short and added if the tape is too long. The correction is calculated using the following expression:

$$C_\ell = \begin{array}{c}\text{Total \# tapes} \\ \text{in measured} \\ \text{distance}\end{array} (\text{L actual} - \text{L standard}).$$

ex.

Lets complete the previous example.

$$C_\ell = \frac{785.62'}{100' \text{ per tape}} (99.92' - 100.00').$$

The quantity 7.8562 is the number of times the tape was laid out in measuring 785.62 feet.

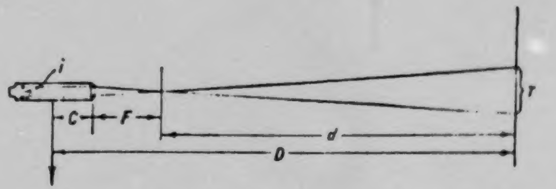
The quantity -.08' is the error in each tape length.

There are several other types of corrections that can be applied in taping. These include corrections for temperature, alignment, support of the tape, and the amount of tension pull applied to the tape ends when readings are taken.

As with any kind of field measurements, there are common mistakes in taping operations. These include faulty tallying of the total number of tape lengths, misreading the tape (particularly 9 and 6 or 8 and 3), mistakes in calling and recording numbers and errors in reading the 1 foot of tape length where fractional parts are marked.

There are several indirect methods of distance measurement. Two methods that utilize basic geometric principles are the stadia method and the subtense method. Both are tacheometric measurements, that is, measurement of a small angle lying opposite a known base in order to obtain a distance. In the stadia method, a fixed angle is used and intercepts on a graduated rod are recorded. In the subtense method, a fixed base is used and the small angle is recorded. Lets consider both methods in more detail.

For stadia distance measurement, we would need a graduated rod and a transit having a telescope with stadia crosswires. All transit telescopes having a + in the telescope, but the stadia crosswires are lines above and below this cross $\overline{\underline{+}}$. The distance between the crosswires is a function of the telescope and is denoted as i. When we look through the telescope at the graduated rod, we will obtain the interval, r, the difference in readings between the upper and lower stadia crosswires. We want to obtain the distance, D, from the plumb line to the graduated rod.



Stadia Principle for Horizontal Sight

For the figure, C is the distance from the plumb line to the telescope lens, F is the focal distance to the graduated rod. Using the principle of similar triangles, $d/r = F/i$. Then $d = r\left(\frac{F}{i}\right)$. Then we can write an expression for the total distance $D = \left(\frac{F}{i}\right)r + (F + C)$. The term (F + C) is not necessary for interior focusing telescopes and is 1.0' or less for most telescopes. The term $\left(\frac{F}{i}\right)$ is called the stadia multiplying factor of the instrument. The procedure for obtaining the multiplying factor, k, is illustrated in the example, and would have to be done in the field before any distance measurement could be accomplished.

ex.

The transit is set up at location A. The transit has an interior focusing telescope and is equipped with stadia crosswires.

From the transit, a distance of 100.0' is measured by taping. The rod is held vertically at this location and r is obtained. A distance of 200.0' is measured by taping, and a second r is obtained. A distance of 400.0' is measured by taping and a third r is obtained. The values of F/i are calculated for each distance. The average value of F/i would then be used for stadia measurements with that transit.

Distance	r	F/i
100.0	2.62	38.17
200.0	5.16	38.76
400.0	10.53	37.99

$k = 38.31$

If this transit is then used to measure a series of three distances, AB, BC, and CD, what total distance is measured?

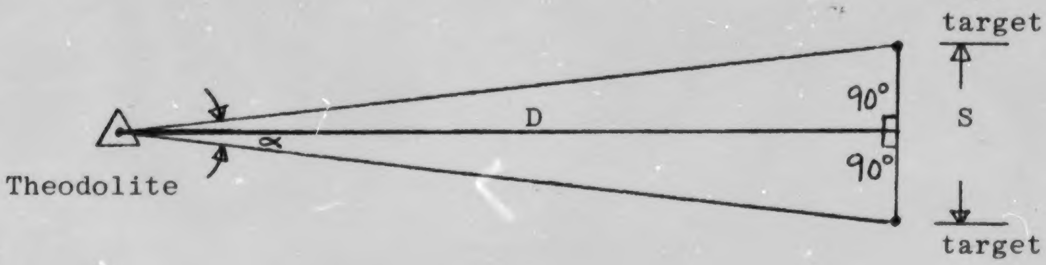
Course	r	D
AB	7.82	299.56
BC	5.42	207.62
CD	12.94	495.69

Total = 1002.87'

As in all field work, it is important to keep in mind the sources of error associated with the stadia method. These include error in reading the rod, error due to the rod not being held straight, value in obtaining the stadia multiplying factor of the instrument, error in the length of the graduated rod, failure to focus the telescope properly, and the natural errors due to climatic conditions. It is important to notice the fact that all readings are obtained using the stadia multiplying factor. If this factor is in error by 2%, then every distance calculated will be in error by at least 2%.

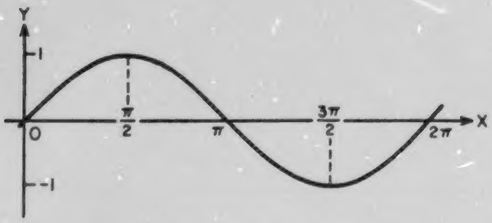
We mentioned before that the subtense method is also a tacheometric method. The subtense method requires measuring the angle lying opposite a fixed base. The fixed base is a subtense bar. A subtense bar is usually 2 meters long and has a target at each end. The bar is mounted on a tripod and is levelled. A theodolite is used to measure the angle between the two targets. Consider the sketch, keeping in mind that we want to find the distance D. The value of S is 2 meters, the length of the subtense bar.

Then $D = \frac{1}{2} S \cdot \cot \frac{\alpha}{2}$.



We have other methods for obtaining horizontal distances which we will consider very briefly. Electromagnetic measurements are used when precise measurements of long distances are needed. Many types of equipment are available but all the methods are based on the principal of measuring wavelengths of a wave from an emitter to a reflector at the point where the distance is desired.

In order to understand these methods, a review of waves is necessary. A typical sinusoidal waveform is shown. The wavelength, λ, is the distance travelled during the period of 1 cycle, that is, the distance between repetitions of the exact same point



$y = \sin x$

in the waveform. The wavelength is dependent on the frequency of the wave, that is, on the number of cycles per unit time. The wavelength is obtained from the expression $\lambda = \frac{V}{f} \quad \frac{\text{velocity of wave propagation}}{\text{frequency}}$.

Lightwave and radio wave velocities vary with atmospheric conditions. The index of refraction, , represents the ratio between wave velocity in a vacuum and wave velocity in atmospheric conditions. $n = \frac{V_o}{V} = \frac{\text{velocity in vacuum}}{\text{velocity in air}} \cong 1{,}003$ for "average" conditions. This equation is used to find for atmospheric conditions.

To measure distance using electronic distance measuring (EDM), a wave is emitted from the originating point, It hits the target and is returned to the originating point. The time is obtained by comparing the phase of the returned beam with that of the original signal. For most EDM devices, this process is handled within the device and a distance readout is provided in digital form.

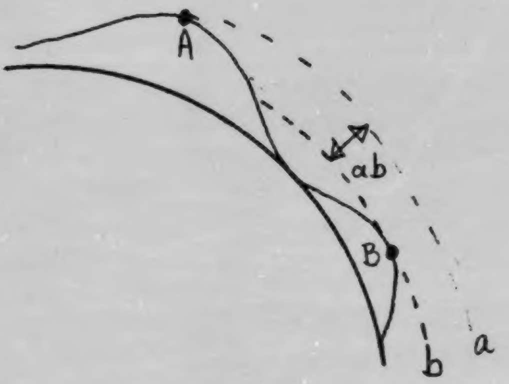
The various types of EDM are different in the types of waves that are emitted. The geodimeter uses light beams. There are also microwave instruments and electro-optical instruments using either visible light or infrared radiation. In all cases, waves are reflected from a variety of nearby objects because the emitted beam is not a line but a cone. The operator must be careful to correctly interpret these multipath reflections so that good results are obtained.

We have seen a variety of distance measuring techniques. The simplest method, pacing, required little equipment but provided the least exact result. As the complexity and expense of the methods and equipment increased, there were corresponding increases in the accuracy of the results. Selection of the "correct" method requires consideration of both accuracy needed and economic limitations.

Determination of Elevation

There are many times when we want to know the elevations of several points. In taping over sloping ground, we need the value of " " in order to apply the slope correction. This value can be obtained by knowing the elevations of the beginning and end points. In laying out railroads and other transportation systems, control of grade changes is required to insure safe travel. In construction, levelling insures that various components of the project fit together as specified in the plans. In this section we will consider the process of levelling as it is commonly used in plane surveying.

A level surface is a surface which is at every point perpendicular to the line on which gravity acts. Levelling is the process of determining the difference in elevation between two points by measuring the vertical distance between the level surfaces, passing through the points. Consider an exaggerated diagram to illustrate these important definitions. The short line ab represents the difference in elevation between A and B while "a" is the level surface through point A and "b" is the level surface through point B. If "c" is also a level surface and all other elevations are expressed in terms of "c", then it would be a datum. Mean sea level is the most commonly used elevation datum.

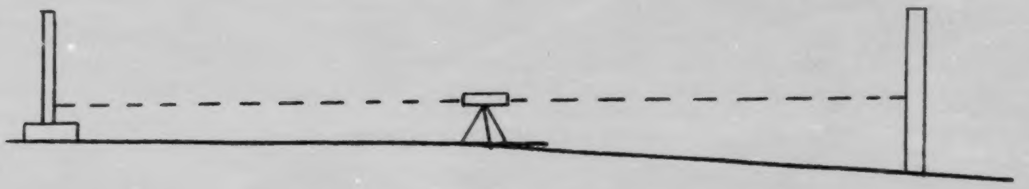


Leveling equipment includes a level, level rod, rod level and turning points. There are many types of levels including the Dumpy level, Tilting level and the Automatic level. Hand levels are also common for such applications as bricklaying. The level consists of a telescope mounted on a tripod. The level, once properly set up, allows the telescope to rotate without any change in the elevation. The level rod is a graduated rod, usually 7 feet long with an extension so that up to 13 feet of elevation change can be read. The rod level is a bubble level that can be attached to the rod so that readings are taken with the rod in the vertical position. Turning points are often stakes or other firm material that can be placed on the ground surface so that no change in rod elevation takes place while readings are being taken.

A few additional definitions are needed before we can actually look at the procedure for leveling. A benchmark (B.M.) is a permanent object of known elevation. Brass disks set in concrete sidewalks are common benchmarks in Texas. The benchmark will be stamped with the name of the agency from whom more

information can be obtained concerning the benchmark. A turning point (T.P.) is a definite, firm object which holds an elevation temporarily in the process of leveling between benchmarks. A rod held on turf is not a suitable T.P. because surface soil or grass are easily compressed during surveying. A backsight (B.S.) is a rod reading taken on a benchmark or turning point of known elevation. It is the vertical distance from the benchmark or turning point to the line of sight of the level telescope. The elevation of the line of sight of the instrument is the height of instrument (H.I.). Then HI = B.S. + Elev. of BM or TP.
A foresight (F.S.) is a rod reading taken on a turning point or other object whose elevation is to be determined. It is the vertical distance from the line of sight of the instrument to the point observed.

Elev (Computed) = HI - F.S.



Lets state briefly the steps in performing a leveling operation.

1. Set up the level at a suitable distance from the benchmark. For a Dumpy level, the bubble

is centered over 1 pair of screws, then over the other. Then sight the telescope on the rod which is held vertically on the BM by the rodman. The bubble is centered again.

2. The B.S. reading is taken and recorded. While the rodman moves to the T.P., the levelman calculates the HI.
3. The T.P. should be approximately the same distance from the level as was the BM. That is, equal distances should be used for backsights and foresights. The telescope is sighted on the T.P. and the bubble is centered. Note that the instrument has not been moved.
4. The F.S. reading is recorded and the elevation of the T.P. is calculated. The rodman remains at the T.P. while the levelman moves the instrument a suitable distance past the T.P.
5. The level is set up and the bubble is centered as before. A B.S. is taken on the last T.P. and is used to calculate a new HI. The Rodman then moves to the next T.P. As the procedure is repeated.

Information from a leveling operation is recorded as the procedure is done in a fieldbook. One type of fieldbook format for leveling is shown below.

DIFFERENTIAL LEVELING

FROM__________ TO__________

STA	B.S.	H.I.	F.S.	ELEV.	Weather Conditions	Party Members
BM_1					Information about B.M.#1	
TP						
TP					As many TP's can be used as necessary to cover the required distance.	
TP						
BM_2					Information about B.M.#2	

We can see by considering the definitions that we can check our results for errors in arithmetic as follows. The sum of all the backsight readings minus the sum of all the foresight readings must total change in elevation between the initial BM and the final location. This arithmetic check is called a page check and is included on the field notes. It does not provide information about errors in taking B.S. or F.S. readings, but only a check on computations. Notice that all computations involve only addition and subtraction.

ex.

Complete the following field notes for differential leveling and perform the page check.

STA	B.S	H.I.	F.S.	ELEV
BM1	9.42			685.43
TP1	8.36		4.62	
TP2	5.07		0.56	
BM2			7.32	

Computations

HI=685.43+9.42=694.85

ELEV TP1=694.85-4.62=690.23
HI=690.23+8.36=698.59
ELEV PT2=698.59-0.56=698.03
HI=698.03+5.07=703.10

ELEV BM2=703.10-7.32=695.78

The completed field notes with page check would look like this:

STA	B.S.	H.I.	F.S.	ELEV
BM1	9.42	694.85		685.43
TP1	8.36	698.59	4.62	690.23
TP2	5.07	703.10	0.56	698.03
BM2			7.32	695.78

$$\Sigma = \begin{array}{r} 22.85 \\ -\ \underline{12.50} \\ 10.35 \end{array} \qquad \Sigma = 12.50 - \begin{array}{r} \underline{685.43} \\ 10.35 \end{array}$$

checks

We will complete our study of leveling by looking at two commonly done types of leveling operations: profile leveling and cross-section leveling. In profile leveling, elevations are found for points at regular intervals along a fixed line or curve. Cross-sections are profiles taken transverse to the centerline of a project. Both have wide application in highway and other construction and are used in the calculation of cut and fill for excavation work.

In profiling, stakes are set at 100 foot intervals called stations. Readings are also taken at plus-stations which are locations between full stations where there is a change in slope.

ex. 68 + 25 is located 25 feet past station 68 or 6825 feet from the 0.0 location.

ex. 35 + 74 is located 74 feet past station 35 or 3574 feet from the 0.0 location.

The procedure is to first establish stations and plus stations along the line. Set up the instrument near the stakes and determine the HI by differential leveling. The instrument should be positioned so that foresights can be taken at several stations or plus stations for each set up. These readings are recorded to the nearest tenth foot and are called Ground Rod Readings (GR). Turning points are used as necessary to establish the HI in order to continue the profiles. The page check is used to check calculations on the elevations for turning points and bench marks. The ground rod readings are listed in a separate column of the field notes in order to facilitate checking the work.

ex. Complete the following field notes for profile leveling:

STA	B.S.	H.I.	F.S.	G.R.	ELEV
BM1	4.25				840.37
4+00				11.7	
5+00				9.2	
5+24				4.9	
6+00				8.5	
TP	3.26		5.92		

Calculations

HI=840.37+4.25=844.62
ELEV=844.6-11.7=832.9
ELEV=844.6-9.2=835.4
ELEV=844.6-4.9=839.7
ELEV=844.6-8.5=836.1
ELEV=844.62-5.92=838.70
HI=841.96

Then the completed set of field notes would be as follows:

STA	B.S.	H.I.	F.S.	G.R.	ELEV
BM1	4.25	844.62			840.37
4+00				11.7	832.9
5+00				9.2	835.4
5+24				4.9	839.7
6+00				8.5	836.1
TP	3.26	841.96	5.92		836.70

Profiles are usually presented graphically plotted on profile paper having a larger vertical scale than horizontal acale. This shows in exaggerated form the variation in the ground surface.

Cross sections are profiles taken on a line perpendicular to the centerline of a project. The two common classes of cross-sections are roadway and borrow pit. Roadway cross sections provide information about the current shape of ground along a roadway, and are performed both prior to and during roadway construction. Borrow pit cross sections are used to determine the volume of select material removed from a gravel quarry or other borrow site.

Roadway cross sections include leveling notes on the left hand page of the fieldbook and cross section notes on the right hand page. Data is collected on both the left and the right of the centerline, usually at points equally spaced, say at 5' intervals. The form of the recorded information is shown:

Left		℄	Right	
Distance	Distance	0	Distance	Distance
$\frac{\text{GR}}{\text{ELEV}}$ = HI - GR	$\frac{\text{GR}}{\text{ELEV}}$ = HI - GR	$\frac{\text{GR}}{\text{ELEV}}$	$\frac{\text{GR}}{\text{ELEV}}$	$\frac{\text{GR}}{\text{ELEV}}$

As many points are used along the transverse line as are necessary to show the roadway shape.

We have seen that leveling operations are more complex insofar as fieldwork is concerned. Care must be taken in leveling the instruments and in preventing any disturbance of the instruments. The rod must be held straight and exactly on the required location. Care must be taken in reading the rod and in accurately recording the data.

The concept of leveling is straightforward and the calculations involve only addition and subtraction. Determination of elevations remains an important aspect of plane surveying.

Traverse

Traverse is one method of establishing horizontal control, that is, of establishing a network of points whose horizontal positions have been accurately determined. Traverse consists of a series of lines whose lengths and directions have been measured. The locations of the connecting points are to be determined.

In order to perform a traverse it is necessary to measure distance as we have already described and angles. Both horizontal and vertical angles can be measured with transits or theodolites. Before looking at the procedure for traverse work, we need to consider units for angular measurements.

Most Americans are familiar with the sexagesimal system for measuring angles where the circumference is divided into 360 parts, each part being 1 degree. Each degree is subdivided into 60 minutes and each minute is subdivided into 60 seconds.

ex. 48° 52' 09" is an angle with 48 degrees, 52 minutes and 9 seconds.

In addition to the sexagesimal system, there is the centesimal system. The circumference is divided into 400 parts, each of which is called a grad. Each grad is divided into 100 centesimal minutes and each minute is divided into 100 centesimal seconds.

ex. $380^g\ 79^c\ 62^{cc}$ is an angle with 380 grads, 79 centesimal minutes and 62 centesimal seconds.

The angle can be easily expressed in decimal form:

$380^g\ 79^c\ 62^{cc} = 380.7962$ grads.

The mil is another unit of angular measurement in which the circumference is divided into 6400 parts, each of which is 1 mil. More commonly used than the mil is the radian. The radian is the angle for which the radius is exactly equal to the subtended arc. $1 \text{ radian} \cong \frac{360}{2\pi} \quad 57.30$.

ex.

Convert $380^g\ 79^c\ 62^{cc}$ to the sexagesimal system.

$$380.7962 \text{ grads} \times \frac{360 \text{ degrees}}{400 \text{ grads}} = 342.717 \text{ degrees}$$

$$.717 \text{ degrees} \times \frac{60 \text{ minutes}}{1 \text{ degree}} = 43.020 \text{ minutes}$$

$$.020 \text{ minutes} \times \frac{60 \text{ sec}}{1 \text{ min}} = 1.2 \text{ seconds}$$

Answer: $342^\circ\ 43'\ 1''$

Horizontal angles are measured in traverse operations. Several different types of horizontal angles can be used. Interior angles are angles inside of a closed traverse. They are demonstrated in figure A. The deflection angle is the angle that any line makes with the extension of the previous line.

Deflection angles must be disignated as Right (R) or Left (L) as shown in figure B. Figure C demonstrates the angle-to-right which is the clockwise angle from the back line to the forward line. Deflection angles and angles-to-right can be used for both closed and open traverses.

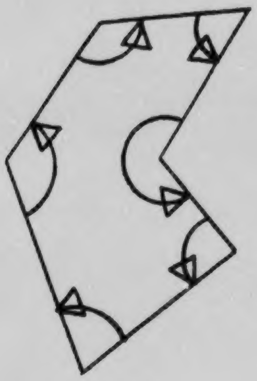


Figure A interior angles

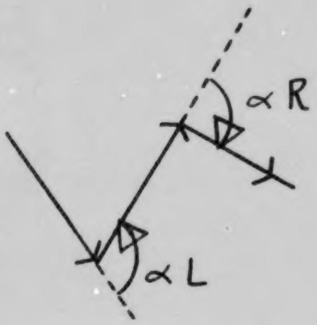


Figure B deflection angles

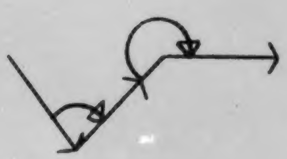
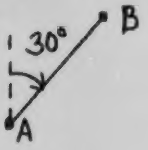


Figure C angle-to-right

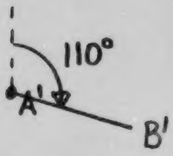
It is necessary to express angles in terms of a meridian which is a reference line to which the directions of all the lines of a survey are referred. An azimuth gives the value of the clockwise angle which the line makes with the north end of a line passing through a point joining the north and south poles of the earth's axis. The bearing is the acute angle (never greater than 90 degrees) which a line makes with a reference meridian.
For every line we can express its orientation with either azimuth or bearing.

ex. Consider the following lines:



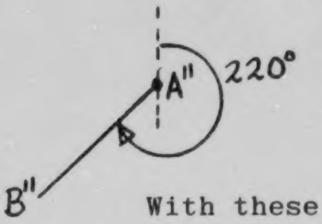
The azimuth of AB is 30 .

The bearing of AB is N30 E.



The azimuth of A'B' is 110 .

The bearing of A'B' is 570 E.



The azimuth of A"B" is 220 .

The bearing of A"B" is S 40 W.

With these definitions in mind, lets consider the parts of the transit. The transit, like the level, has a leveling head that supports the transit on the tripod and provides the means of leveling the instrument. The instrument must be carefully leveled before any angles can be measured. The lower plate consists of a hollow spindle fitting the socket in the leveling head. It also contains a graduated horizontal circle. The upper plate contains the verniers for reading angles, the standards supporting the telescope the bubble tubes and a compass. The telescope contains an objective lense, the eyepiece which reinverts and enlarges the imare from the objective lense and cross-wires for sighting on the object to be observed.

A backsight is a sight taken with the telescope on a point, usually the last preceding point. A foresight is a sight taken to fix the direction of a line. Angles are read when the vernier is set to zero on the backsight, the desired angle is turned, and the angle is read on the vernier.

We are now ready to consider a closed traverse, that is, a traverse that begins and ends at points with known horizontal positions. A station in a traverse is a point at which the transit is set up and the angular change in direction is measured. The station is typically marked with a stake having a thumbtack or mark to indicate the exact point. Distance measurements are made by taping. Corrections for slope, temperature, sag and other systematic errors are applied if necessary. Angular measurements can use deflection angles, angle-to-right or interior angles. The angle-to-right is particularly easy to do in the field. To do this, take a sight on the last station occupied and turn a clockwise angle to the next traverse point.

Deflection angles provide a method of checking angles since the sum of the deflection angles for a looped traverse should equal 360°. Interior angles can also be used to check the angles since the sum of the interior angles for an "n" sided closed figure should equal $(n - 2)(180^\circ)$. The angular error of closure is the difference between the actual sum of the interior angles and the theoretical sum $(n - 2)(180^\circ)$. Before any other calculations are made, the angular error is distributed by dividing it equally among the angles measured.

ex.

A loop traverse is performed that yields a triangular area. The bearing of AB is N 80° 00' W. The following distances and interior angles were obtained. The first angle is at point A.

Course	Distance (ft.)	Interior ∢
AB	500.01	47° 15' 00"
BC	399.93	21° 38' 45"
CA	175.24	111° 06' 00"

(1) Calculate the angular error of closure and distribute.

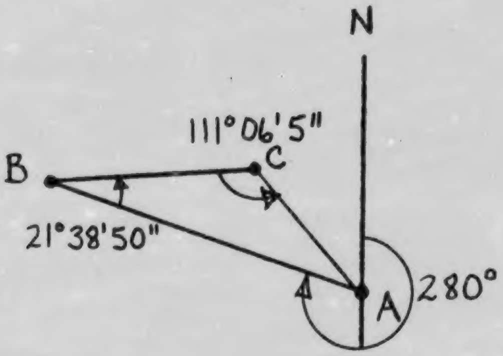
The sum of the interior angles should equal 180 00'00"
The actual sum = 179° 59' 45". The actual sum is too small, so the correction must be added to each interior angle.

Angular error of closure = 15". Correction per ∢ = 5".

Corrected Interior ∢A = 47° 15' 5"
∢B = 21° 38' 50"
∢C = 111° 06' 5"

(2) Sketch the figure and find the bearings of BC and CA.

Notice that the bearing of AB is N 80° 00' N.
Thus the azimuth of AB is 280°.



To get bearing of BC:

$$\begin{array}{r} 179^\circ\ 59'\ 60'' \\ -\ \underline{21^\circ\ 38'\ 50''} \\ 158^\circ\ 21'\ 10'' \end{array}$$

158° 21' 10" = change in azimuth between AB and BC

158° 21' 10"
+ 280°
438° 21' 10" would be azimuth
- 360° of BC except>360°,
78° 21' 10" azimuth of BC.
N 78° 21' 10" bearing of BC.

To get bearing of CA.

179° 59' 60"
- 111° 06' 05"
68° 53' 55" = change in azimuth
between BC and CA.

68° 53' 55"
+ 78° 21' 10"
146° 74' 65" = 147° 15' 05" azimuth of CA

But the bearing cannot be greater than 90°.

azimuth of 146° 15' 05" = bearing S 32° 44' 55" E.

We will continue this example after discussing more traverse computations. A latitude is the projection of a line on the reference meridian. The latitude is equal to the product of the length and the cosine of the bearing angle. The value is positive if the bearing is northerly and negative if the bearing is southerly. A departure is the projection of a line on the east - west line perpendicular to the reference meridian. The value is obtained using the formula:

departure = length x sine bearing angle.

The departure is positive if the bearing is easterly and negative if the bearing is westerly. For any loop traverse, the algebraic sum of the latitudes and that of the departures should be zero because the survey begins and ends on the same point. This linear error is balanced so that closure is obtained.

The procedure for balancing is called the Compass Rule and consists of making a proportion of the total correction for each course according to its portion of the total length of the traverse. For any course XY, we use the expression:

$$\frac{\text{Correction to XY latitude}}{\Sigma \text{ latitudes}} = \frac{\text{Length of Course XY}}{\text{Length of Traverse}}$$

$$\frac{\text{Correction to XY departure}}{\Sigma \text{ departures}} = \frac{\text{Length of Course XY}}{\text{Length of Traverse.}}$$

The proper sign must be used so that the corrected sum of the latitudes is equal to zero and the corrected sum of the departures is equal to zero.

ex. (continued)

(a) Calculate all latitudes and departures. Find the error in the latitudes and the error in the departures.

	LATITUDES		DEPARTURES	
Course	North	South	East	West
AB	86.83			492.41
BC	80.74		391.70	
CA		147.39	94.80	

Sum of Latitudes = 86.83 + 80.74 - 147.39 = 20.18 = error in lat.

Sum of Departures= 391.70 + 94.80 - 492.41 = -5.90 = error in dep.

(b) Distribute the error according to the Compass Rule.

$$\frac{\text{Correction AB Lat}}{20.18} = \frac{500.01}{1075.18} \qquad \frac{\text{CABD}}{5.90} = \frac{500.01}{1075.18}$$

CABL = 9.385 CABD = 2.74

$$\frac{\text{C BC L}}{20.18} = \frac{399.93}{1075.18} \qquad \frac{\text{CBCD}}{5.90} = \frac{399.93}{1075.18}$$

CBCL = 7.51 CBCD = 2.19

$$\frac{\text{CCAL}}{20.18} = \frac{175.24}{1075.18} \qquad \frac{\text{CCAD}}{5.90} = \frac{175.24}{1075.18}$$

CCAL = 3.29 CCAD = 0.96

Because the north latitudes are bigger than the south, the latitude corrections are subtracted for AB and BC, and added for CA.

Because the east departures are smaller than the west, the departure corrections are added for BC and CA and subtracted for AB.

LATITUDES			DEPARTURES	
Course	North	South	East	West
AB	77.45			489.67
BC	73.23		393.90	
CA		150.68	95.77	
Σ =	150.68	150.68	489.67	489.67

Note that the sum of the latitudes = 0.0 and the sum of the departures = 0.0

Once the latitudes and the departures are balanced, the coordinates of the points can be found. Assume the Y_A is known or assumed y - coordinate of point A. Then we can find Y_B, the y - coordinate of point B: $Y_B = Y_A$ + Latitude of AB. Similarly to find the x - coordinate of point B: $X_B = X_A$ + departure of AB. Balanced latitudes and departures must be used in calculating coordinates.

ex. (continued)

If the coordinates of point A are (1000,1000), find the coordinates of points B and C.

Point B: $X_B = X_A - 489.67 = 1000 - 489.67 = 510.33$

$Y_B = Y_A + 77.45 = 1000 + 77.45 = 1077.45$

B coordinates = (510.33, 1077.45)

Point C: $X_C = X_B + 393.90 = 510.33 + 393.90 = 904.23$

$Y_C = Y_B + 73.23 = 1077.45 + 73.23 = 1150.68$

C coordinates = (904.23, 1150.68)

Calculation of coordinates was the goal of traverse. This information may be used to establish a control system. It can also be used to calculate areas although we will not cover this procedure. Traverse operations are often done as part of deed checks in land sales and are important in many civil engineering applications. Traverse computations involve both simple arithmetic and trigonometric functions because of the angular measurements involved. In addition, linear ratios are used in balancing the latitudes and departures.

It is hoped that this brief discussion of three primary plane surveying measurements will provide insight into applications of mathematics in the field of surveying.

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INTRODUCTION TO COMPUTER APPLICATIONS

Number Systems

Modern computer systems perform their operations using binary numbers. Binary (base 2) is a system of numeration using only the digits 0 and 1. Numbers stored in computer memory are usually in groups of 8 or 16 binary digits.

A partial listing of memory containing only binary digits is cumbersome. Converting binary digits to hexadecimal (base 16) reduces the number of digits by a factor of 4. The table provided is helpful when converting binary to hexadecimal, or hexadecimal to binary. Equivalent numbers as we know, are listed in the "Decimal" column, which is base 10.

It is important to know the procedure for converting to hexadecimal. This numeration system of data can be easily input, or output by a computer programmer.

TABLE Hexadecimal → Binary → Decimal Value

Hexadecimal	Binary	Decimal Value
0	0000	0
1	0001	1
2	0010	2
3	0011	3
4	0100	4
5	0101	5
6	0110	6
7	0111	7
8	1000	8
9	1001	9
A	1010	10
B	1011	11
C	1100	12
D	1101	13
E	1110	14
F	1111	15

Converting Binary to Hexadecimal

Example:

Convert 1010110010111001. to hexadecimal.

Solution:

Start with the binary point and separate the binary digits into groups of 4. Then, convert each group of 4 to its corresponding hexadecimal and connect.

```
1010     1100     1011     1001.
 \/       \/       \ /      \/
  A        C        B        9        =  ACB9
```

PROBLEMS

Convert the following binary numbers to hexadecimal:

1) 110011.

2) 1010011.

3) 10001010.

4) 11111111.

Answers:

(1) 33, (2) 53, (3) 8A, (4) FF

Converting Hexadecimal to Binary

Example:

Convert the hexadecimal number AB to binary.

Solution:

$$\begin{array}{ccccc} A & & B & & \\ \downarrow & & \downarrow & & \\ 1010 & & 1011 & = & 10101011 \end{array}$$

Example:

Convert the hexadecimal number 89CD to binary.

Solution:

$$\begin{array}{ccccccc} 8 & & 9 & & C & & D & & \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\ 1000 & & 1001 & & 1100 & & 1101 & = & 1000100111001101 \end{array}$$

PROBLEMS

Convert the following hexadecimal numbers to binary:

1) AB25.

2) 26A3.

3) 8A29.

4) 4281.

Answers:

(1) 1010101100101000

(2) 10011010100011

(3) 1000101000101001

(4) 100001010000001

Processing Data

It should be noted that the binary system used in a computer is also known as a "code with two digits, 0 and 1". Data is processed by timed electrical spots. These electric charges, or magnetic spots of electricity, represent the digit 1. No electric charge, or no spot is represented by the digit 0. This code is also called an "on/off", or "yes/no" system.

Binary information is stored in the computer memory. Minicomputers can hold usually 4,000 to over 32,000 characters in the main storage. Microcomputers, having a smaller central processing unit, can retain about 1,000 up to 32,000 characters in the main (primary) storage. When discussing memory capability, the letter "K", which represents "1,000" is frequently used. For example, 32K denotes 32,000 character memory.

Computers process data based simply on the presence, or absence of electrical charges. It is easy to understand why the digits 0 and 1 are preferred to represent these electrical pulses. Therefore, the numeration system of base 2 (binary) is used because the only digits in base 2 are 0 and 1.

Of course, binary and hexadecimal forms of data are not the only codes used by computers. An eight-digit system, called base 8 or octal, is sometimes utilized. There is also the alpha-numeric system. However, numbers expressed in this particular code cannot be controlled arithmetically. Where it is possible to convert a numeral as we know in base 10, to an equivalent value in another base, the characters in alpha-numeric will not allow this.

Though the computer is processing information in base 2, 8, or 16, the programmer must also be familiar with the bases.

Base 2 (Binary)

In the binary system, only two digits are used: 0 and 1. Since we express numerals in base 10, a value of "nine" is wirtten 9.

Base 2 can have a value of "nine", but since 0 and 1 are the only digits in the system, how would the value "nine" be written?

One method of converting base 10 to base 2, is beyond the scope of this lesson. However, a simple way is as follows:

To convert a base 10 number to a binary (base 2) number, continually divide the number by 2, until there is no remainder.

9 2 = 4 + Remainder 1

4 2 = 2 + Remainder 0

2 2 = 1 + Remainder 0

1 2 = 0 + Remainder 1

The new binary number is composed of the remainders, read from bottom to top:

Base 10	Base 2 (Binary)
9	1001

The programming language chosen to process data should be of a type which allows options as to what codes are used. Assembler language is the most flexible. A programmer has the option of data input and output in several coded forms, including binary, hexadecimal, and alphanumeric. Because of the flexibility and varied code options, Assembly language is usually difficult to comprehend and learn.

Basic - An Introduction

An easy computer language to learn is BASIC (Beginner's All-purpose Symbolic Instruction Code). The language is structured to permit the programmer to use flexibility when entering data. Flow-chart instructions and resulting print-outs are easy to read and understand.

BASIC is considered an "interpretive language". This means, a programmer must and can interact with the program as it is being executed by the computer. BASIC language is utilized, along with other interpretive languages, in computer assisted instruction and home computers; from calculating bank statements to programming the lights to come on at night while on vacation.

Applied Problems Using BASIC

The purchase of a new home or automobile is an important event in our lives. When borrowing money for large purchases, it is important to know how monthly payments are computed. Also, how much of each payment is applied to paying off the interest (the price for borrowing money).

The formula for determining the payment for a loan, with monthly payments for a fixed number of years, at a fixed interest rate is:

$$\text{Monthly Payment} = \frac{P \cdot R \cdot (1 + R)^n}{(1 + R)^n - 1} \text{, where}$$

P → Amount Financed
(Purchase price - Down Payment)

R → Interest Rate per Compounding Period
(APR/12 → APR ÷ 12)

n → Number of monthly payments.

We will state a typical loan problem and solve first by hand calculations; then, a computer program in BASIC will be shown with the printed answer.

Problem I:

A new car is purchased at $9,200 with a down payment of $900. The interest rate is 17% and the duration of the loan is 3 years. What is the monthly payment?

Solution:

P → Purchase Price - Down Payment
P = $9200 - $900
P = $8300

$R \rightarrow$ Interest Rate per Compounding Period

$R = .17/12$

$R = 0.01417$

$n \rightarrow$ Number of Monthly Payments

$n = 3 \text{ years} \cdot 12 \text{ months}$

$n = 36$ monthly payments

Formula

$$\boxed{\text{Monthly Payment} = \frac{P \cdot R \cdot (1 + R)^n}{(1 + R)^n - 1}}$$

$$\text{Monthly Payment} = \frac{(\$8300) \cdot (0.01417) \cdot (1 + 0.01417)^{36}}{(1 + 0.01417)^{36} - 1}$$

Monthly Payment = $295.93

A hand calculator would be necessary in order to find the value of $(1 + 0.01417)^{36}$. However, if we structured a BASIC program to calculate the monthly payment, it would look like this:

PROBLEM I

```
5 REM   CALCULATE MONTHLY PAYMENTS
10 PRINT:PRINT "ENTER AMOUNT FINANCED"
20 PRINT "(PURCHASE PRICE MINUS DOWN PAYMENT)"
30 INPUT P
40 PRINT:PRINT "ENTER NOMINAL PERCENTAGE RATE IN DECIMAL"
50 INPUT R
60 LET R=R/12
70 PRINT:PRINT "ENTER DURATION OF LOAN IN YEARS"
80 INPUT N
90 LET N=N*12
100 LET A=(1+R)^N
110 LET A=A/(A-1)
120 REM    .005 REQUIRED TO ROUND UP TO NEAREST CENT
130 LET A=A*R*P+.005
140 PRINT:PRINT"MONTHLY PAYMENT =";USING"$$###.##";A;
150 PRINT " FOR"N"MONTHS"
160 END

RUN

ENTER AMOUNT FINANCED
(PURCHASE PRICE MINUS DOWN PAYMENT)
? 8300

ENTER NOMINAL PERCENTAGE RATE IN DECIMAL
? .17

ENTER DURATION OF LOAN IN YEARS
? 3

MONTHLY PAYMENT = $295.92 FOR 36 MONTHS
```

Now that we know the monthly payment, it would be interesting to know how much of the monthly payment is applied to the interest, and how much of the monthly payment is applied to the principal sum (amount financed).

To answer these questions, an amortization schedule is required. To construct a schedule by hand calculations, especially for our auto loan problem, would be lengthy and tedious. However, it is possible.

The formula for the monthly balance, each month of our 3 year loan, is:

$$S = S + (S \cdot R) - \text{Monthly Payment}$$

S → Loan Balance

R → Interest Rate/12

Constructing an amortization schedule for the auto loan for $8300 at 17% for 3 years:

S → Loan Balance = $8,300

R → Interest Rate/12 = .17/12 = 0.01417

P → Monthly Payment = $295.93

Therefore, after payment #1, the loan balance is:

$$S = S + (S \cdot R) - \text{Monthly Payment}$$

$$S = \$8{,}300 + (\$8{,}300 \cdot 0.01417) - \$295.93$$

$$S = \$8{,}300 + (\$117.61) - \$295.93$$

$$S = \$8{,}417.61 - \$295.93$$

$$S = \$8{,}121.68$$

After payment #2, the loan balance would be:

$$S = S + (S \cdot R) - \text{Monthly Payment}$$

$$S = \$8,121.68 + (\$8,121.68 \cdot 0.01417) - \$295.93$$

$$S = \$8,121.68 + \$115.08 - \$295.93$$

$$S = \$8,236.76 - \$295.93$$

$$S = \$7,940.83$$

This procedure would have to be done after each of the 36 monthly payments. But utilizing a BASIC program, the amortization schedule is computed in tabular form (per written commands) in a matter of seconds. The program is:

AMORTIZATION SCHEDULE

```
5 REM      AMORTIZATION SCHEDULE FOR LOAN WITH
10 REM     FIXED MONTHLY PAYMENTS AND NOMINAL INTEREST RATE
20 PRINT:PRINT "ENTER AMOUNT FINANCED"
30 INPUT S
40 PRINT "ENTER NOMINAL INTEREST RATE IN DECIMAL"
50 INPUT R
60 LET R=R/12
70 PRINT:PRINT "ENTER MONTHLY PAYMENT"
80 INPUT M
90 LET N=0
100 LPRINT "NUMBER","PRINCIPAL","INTEREST","BALANCE"
110 LPRINT N,USING "####.##       ";P,I,S
120 LET N=N+1
130 LET I=S*R
140 LET S=S+I
150 IF M-S>0 THEN M=S
160 LET P=M-I
170 LET S=S-M
180 LPRINT N,USING "####.##       ";P,I,S
190 IF S>0 GOTO 120
200 END

RUN

ENTER AMOUNT FINANCED
? 8300

ENTER NOMINAL INTEREST RATE IN DECIMAL
? .17

ENTER MONTHLY PAYMENT
? 295.92
```

NUMBER	PRINCIPAL	INTEREST	BALANCE
0	0.00	0.00	8300.00
1	178.34	117.58	8121.66
2	180.86	115.06	7940.80
3	183.43	112.49	7757.37
4	186.02	109.90	7571.35
5	188.66	107.26	7382.69
6	191.33	104.59	7191.36
7	194.04	101.88	6997.32
8	196.79	99.13	6800.53
9	199.58	96.34	6600.95
10	202.41	93.51	6398.54
11	205.27	90.65	6193.27
12	208.18	87.74	5985.08
13	211.13	84.79	5773.95
14	214.12	81.80	5559.83
15	217.16	78.76	5342.68
16	220.23	75.69	5122.44
17	223.35	72.57	4899.09
18	226.52	69.40	4672.58
19	229.73	66.19	4442.85
20	232.98	62.94	4209.87
21	236.28	59.64	3973.59
22	239.63	56.29	3733.96
23	243.02	52.90	3490.94
24	246.47	49.45	3244.48
25	249.96	45.96	2994.52
26	253.50	42.42	2741.02
27	257.09	38.83	2483.93
28	260.73	35.19	2223.20
[illegible]	264.42	31.50	1958.78
[illegible]	268.17	27.75	1690.61
31	271.97	23.95	1418.64
32	275.82	20.10	1142.81
33	279.73	16.19	863.08
34	283.69	12.23	579.39
35	287.71	8.21	291.68
36	291.68	4.13	0.00

FORTRAN - An Introduction

The name FORTRAN was derived from its beginning purpose, Formula Translation. Developed in the late 1950's, FORTRAN was designed to be an easier language for scientists than the existing assembly language. Unlike BASIC, which is an interpretive language (programmer interaction), FORTRAN is referred to as a "compiler" language. This means if the computer has a FORTRAN compiler in place, the programmer can input instructions (a program in source code) and the computer translates the instructions to machine language. Results of the program are then furnished on a computer print-out.

Program Source Code

BASIC has a list of rules and instructions that a programmer must use during computer interaction. The FORTRAN language also has rules and procedures necessary in accordance to compiler.

FORTRAN "commands" are written by the programmer on a FORTRAN Coding Sheet*, as illustrated. Columns 1 - 5 are used for identification purposes. The computer will execute instructions based on information in the other columns, and not what is written in columns 1 - 5.

Column 6 is for "continuation"; denoted usually by a C. If more information is necessary on a card than columns provide spaces, another card can be used, as long as the computer "reads" a C. For example, a lengthy scientific formula may have to be written on two cards, or two lines on the coding form. The second card must have a character in column 6 to denote the continuation of the formula which began on the first card, or line.

---

* Permission to reprint by IBM Corporation

Columns 7 - 72 are spaces in which the programmer writes FORTRAN instructions. Remember, this is where the FORTRAN compiler "translates" the instructions to machine language in order for program to be executed. As stated before, it is because of this procedure that FORTRAN is referred to as a compiler language.

The last columns of the FORTRAN coding form are 73 - 80. These spaces are mostly used to provide instruction sequence identification for the programmer.

IBM FORTRAN Coding Form

GX28-7327-6 U/M 050**
Printed in U.S.A.

PROGRAM
PROGRAMMER
DATE
PUNCHING INSTRUCTIONS
GRAPHIC
PUNCH
PAGE OF
CARD ELECTRO NUMBER*

C FOR COMMENT	STATEMENT NUMBER	CONT.	FORTRAN STATEMENT	IDENTIFICATION SEQUENCE
			[illegible]	
			[illegible]	
			[illegible]	
			[illegible]	

*A standard card form, IBM electro 888157, is available for punching statements from this form

**Number of forms per pad may vary slightly

Fortran Functions

The following table is provided from Source: ANS FORTRAN, ANSI X3.9-1966, Table 4, Page 25. The table provides an overview of the basic functions utilized in engineering and scientific data, with the corresponding symbolic names in FORTRAN language.

BASIC EXTERNAL FUNCTION	DEFINITION	NUMBER OF ARGUMENTS	SYMBOLIC NAME	TYPE OF	
				ARGUMENT	FUNCTION
Exponential	e^a	1	EXP	Real	Real
		1	DEXP	Double	Double
		1	CEXP	Complex	Complex
Natural logarithm	$\log_e (a)$	1	ALOG	Real	Real
		1	DLOG	Double	Double
		1	CLOG	Complex	Complex
Common logarithm	$\log_{10} (a)$	1	ALOG10	Real	Real
			DLOG10	Double	Double
Trigonometric sine	sin (a)	1	SIN	Real	Real
		1	DSIN	Double	Double
		1	CSIN	Complex	Complex
Trigonometric cosine	cos (a)	1	COS	Real	Real
		1	DCOS	Double	Double
		1	CCOS	Complex	Complex
Hyperbolic tangent	tanh (a)	1	TANH	Real	Real
Square root	$(a)^{1/2}$	1	SQRT	Real	Real
		1	DSQRT	Double	Double
		1	CSQRT	Complex	Complex
Arctangent	arctan (a)	1	ATAN	Real	Real
		1	DATAN	Double	Double
	arctan (a_1/a_2)	2	ATAN2	Real	Real
		2	DATAN2	Double	Double
Remaindering*	$a_1 (\bmod a_2)$	2	DMOD	Double	Double
Modulus		1	CABS	Complex	Real

Mathematical Operators in FORTRAN

The preceding table illustrated the FORTRAN symbols used for mathematical functions. Simple Mathematical operations in FORTRAN are listed in the next table, with corresponding BASIC symbols:

FUNCTION	SYMBOL(S)	BASIC	FORTRAN
Addition	+	+	+
Subtraction	-	-	-
Multiplication	x, •, ( )	*	*
Division	÷, /	/	/
Exponential	Expressed as a power	**	**

Further Explanation on Mathematical Operators

The chart provided below illustrates additional math symbols, their function, and a comparison between their corresponding symbols in BASIC and FORTRAN.

SYMBOL	FUNCTION	BASIC	FORTRAN
>	Greater Than	>	.GT.
<	Less Than	<	.LT.
=	Equal To	=	.EQ.
≠	Not Equal To	><	.NE.
≥	Greater Than or Equal To	>=	.GE.
≤	Less Than or Equal To	<=	.LE.

MACHINE PARTS INSPECTION

Anodizing Aluminum Finishes

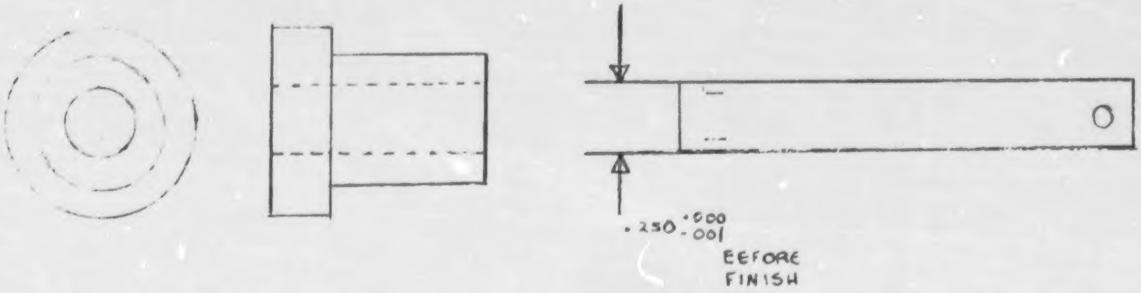
Exterior aluminum finishes are frequently anodized to extend its usefullness and increase its commercial value. The natural oxide film on aluminum which is about 0.0000005 inch thick is increased by anodizing to a thickness between 0.0001 and 0.0003 inch, giving the metal surface a glaze-like quality as if it had been lacquered. Anodic film can be given a wide range of brilliant metallic colors, and yet remain highly resistant to weather, corrosion, abrasion and wear.

Maximum thickness, porosity, hardness and other properties of anodic films depend upon the aluminum alloy and temper, the electrolyte, electrolyte concentration, voltage, current density, temperature, agitation, and similar factors. Coating thickness variations depend primarily on alloy and length of anodizing time, all else being controlled by process parameters. Often, on close tolerance parts which are to be anodized, allowance must be made for dimensional growth of the coating. Only a portion of the anodic coating is dimensional increase, the remainder is penetration.

The following table provides typical dimensional growth properties of various anodic coatings on common aluminum alloys.

Anodize Type	Purpose	Average Thickness	Dimensional Growth per Surface		
			2024AL	5052AL	6061AL
sulphuric acid	electrical resistance exterior finish	0.0001	0.0001	0.0001	0.0001
chromic acid	corrosion resistance; base for paint systems	0.0007	0.0003	0.00035	0.00025
hard (proprietary)	abrasion resistance; corrosion resistance	0.0025	0.0013	0.0015	0.0014

Consider an electronic assembly in which a bushing and shaft 6061-T6 is hard anodized to increase abrasion resistance with the shaft which it contains. Calculate the minimum and maximum inside diameter of the bushing, prior to anodizing, to assure clearance 0.0005 to 0.001 in the assembly after anodizing both the shaft and the bushing.



Shaft $.250 \, {}^{+\,000}_{-\,001}$

$.250 + 2(.0014) = .250 + .0028 = .2528$

$\therefore$ shaft finished dia = $.2528 \, {}^{+\,000}_{-\,001}$ or .2518/.2528

Min ID of bushing @ .2518 shaft = $.2518 \, {}^{+\,.0005}_{+\,.0010} = .2523$

Max ID of bushing @ .2528 shaft = $.2528 \, {}^{+\,.0005}_{+\,.0010} = .2538$

Bushing finished dia = .2523/.2538

Bushing dia before anodize = .2523/.2538 - 2(.0014)

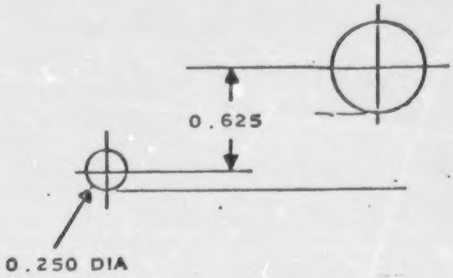
= .2523/.2538 - .0028

= .2495/.2510

Inspection Procedures

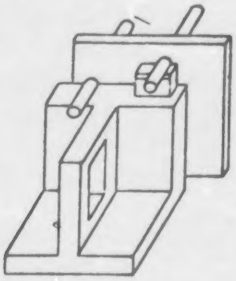
Determining the most convenient, yet sufficiently accurate technique necessary to inspect features of a part is typically as significant a task as the actual measurement.

Consider, for example, an electronic assembly containing a gear assembly housing which as both close tolerance hole diameters as well as hold locations. In the figure below, both positional dimensions have a tolerance requirement of +0004, -0000.

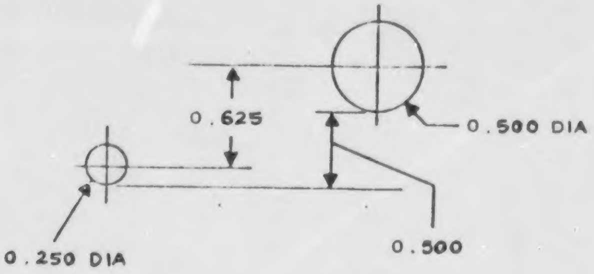


Using precision gage blocks in an angle plate setup on a surface plate, we can determine the dimension between the two holes with a relatively simple technique using precision plug gages (drill or blank) as pins.

In surface plate work, measuring is always perpendicularly upward from the horizontal plane of the surface plate which is the general reference surface. From the sketch below and the height reference gage dimensions provided, determine the acceptability of the dimension between the two holes.



By adding half the diameter of the 0.2500 diameter hole to the 0.6250 dimension and subtracting half the 0.500 diameter hold, we determine that the precision gage block build up required to "level" the two holes (since they are in different planes) is 0.500.



Height Reference Gage Measured Dimension to Top of 0.2500 pin=9.2501
Height Reference Gage Measured to Top of 0.5000 pin = 10.0007

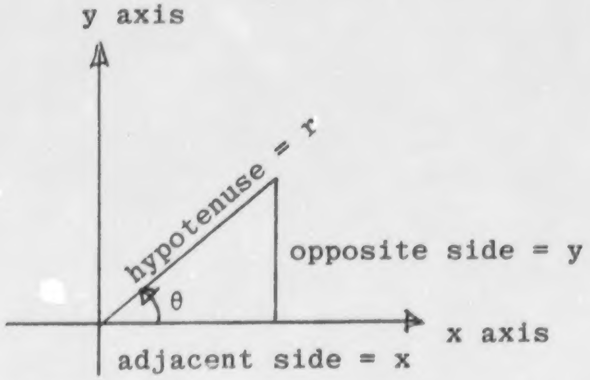
Assuming no error in setup (pins, angle, plate, gage, blocks, measurement) the calculated hole to hole centerline distance is:

10.0007	top of 0.5000 pin
- 0.2500	half the 0.5000 hole diameter
9.7507	℄ of 0.5000 hold
9.2501	top of 0.2500 pin
- .1250	half the 0.2500 hole diameter
9.1251	℄ of 0.2500 hole
9.7507	
- 9.1251	
0.6256	0.6256 exceeds the 0.6250 $^{+0004}_{-0000}$ required dimension by 0.0002

Observation

Before discussing the concept of COSINE ERROR, we should review the function of cosine:

$$\cos\theta = \frac{\text{adjacent side}}{\text{hypotenuse}}$$


y axis
hypotenuse = r
opposite side = y
θ
x axis
adjacent side = x

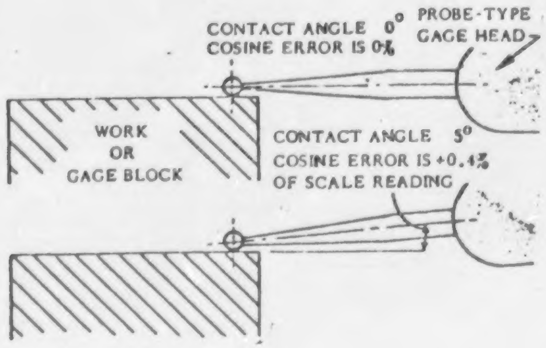
In this case, the cosine of angle θ, is equal to the opposite side of θ, which is y, divided by the adjacent side of θ, which is x.

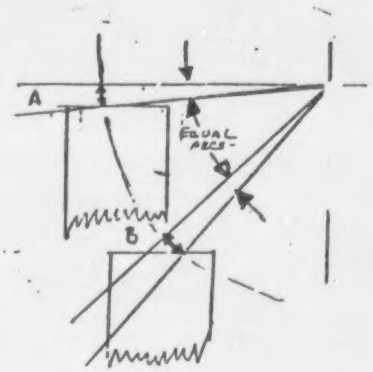
Cosine Error

When using test (dial) indicators, inspectors and machinists alike are concerned with reliable measurements and must become well acquainted with the possibility of "cosine error."

As shown in the figure below, as the angle of the indicator tip increases, the amount of vertical movement that is required to encompass the same arc will increase. A 0.010 inch step at "B" will result in a much larger arc than at "A".

The length of the arc is directly related to the movement of the indicator hand and the indicator is designed to read correctly when the indicator point or probe is in a horizontal position. The angle that the probe deviates from horizontal will result in a proportional error.

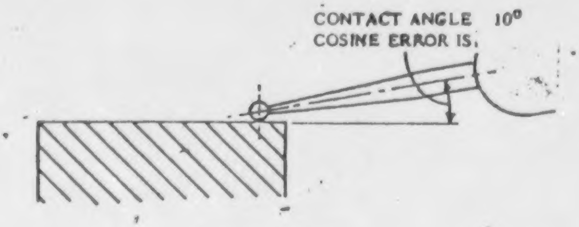

CONTACT ANGLE 0°
COSINE ERROR IS 0%
PROBE-TYPE GAGE HEAD
WORK OR GAGE BLOCK
CONTACT ANGLE 5°
COSINE ERROR IS +0.4% OF SCALE READING


A
EQUAL ARCS
B

Further explaining how to determine cosine error

After calculation of the contact angle, the result is subtracted from 1.000, and then multiplied by 100%.

For example:


CONTACT ANGLE 10°
COSINE ERROR IS

By protractor measurement,

Contact Angle is 10 .

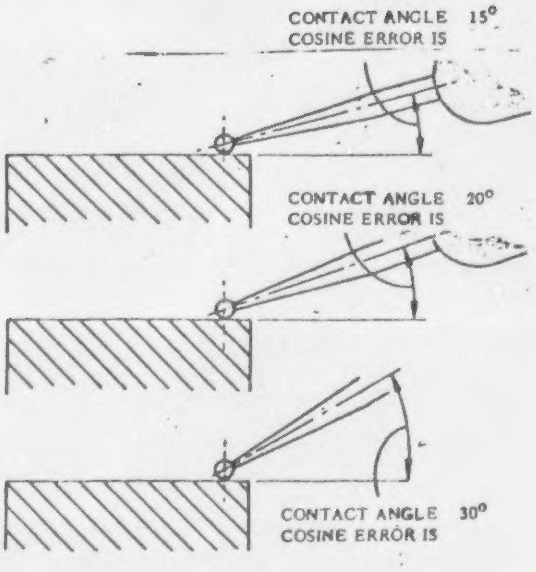
Therefore, cos 10   =

COSINE ERROR = 1.000 - 0.985

= 0.015 · (100%)

= 1.5%

Calculate the following cosine errors for each given contact angle.


CONTACT ANGLE 15°
COSINE ERROR IS
CONTACT ANGLE 20°
COSINE ERROR IS
CONTACT ANGLE 30°
COSINE ERROR IS

SOLUTIONS

cos 15°=.966
cosine error=3.4%
cos 20°=.940
cosine error=6.0%
cos 30°=.866
cosine error=13.4%

True Position Dimensioning

True position dimensioning systems have been used extensively in the aircraft and automotive industries for the past 15 years. More recently, other industries have become aware of its advantages. True position dimensioning and tolerancing is a method of specifying location tolerances for features (holes, slots, keyways, etc.) from a specified theoretically exact location (true position). Its use allows the use of basic dimensions without accumulation of tolerances. When interchangability is a prime consideration, a major advantage of true position dimensioning is noted in that all available tolerance is easily and clearly specified. Other advantages include:

1. Permits the use of maximum tolerance while still maintaining one hundred percent interchangability.
2. Establishes a more uniform way of expressing tolerances.
3. Ensures interchangeability without resorting to small tolerances as must be done in the coordinate system.
4. In the true position tolerance zone there is a large area in which to locate the center.
5. Desired tolerances are arrived at more easily and quickly.
6. Tolerancing of complex parts is simplified.
7. Locations are expressed more adequately than by any other method.
8. There is no accumulation of tolerances.
9. Interpretation of tolerances is made easier for both manufacturing and inspection.

10. True position permits the use of functional gages which simulate the mating part.

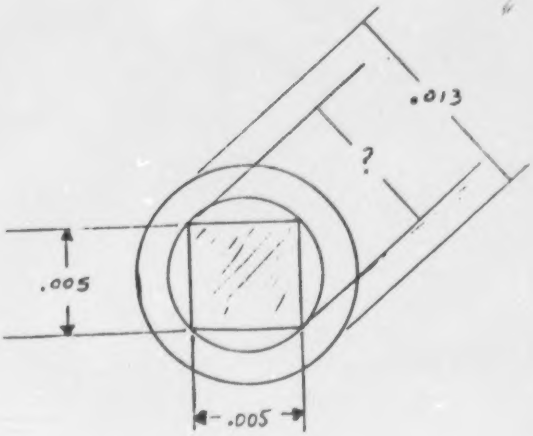
11. Parts previously rejected by inspection to the coordinate system may now prove to be acceptable by the use of true position dimensioning.

Maximum Material Condition Tolerance

As can be seen below, a bonus tolerance over the standard coordinate dimensioning system is obtained from the simple fact that true position dimensioning allows a cylindrical tolerance zone rather than a square tolerance zone.

When the .250±.003 holes are at minimum material condition, i.e.. /253 inch diameter, we can see that the .005 square tolerance zone of the coordinate system can actually be increased to .013 inch.

Calculate the total tolerance available when the .250±.003 holes is at maximum material condition (mmc) or .247 inch diameter.



Ans: Solve for hyp of triangle

$A^2 + B^2 = C^2$

$.005^2 + .005^2 = C^2$

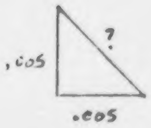
$.000025 + .000025 = C^2$

$.000050 = C^2$

$C = \sqrt{.000050}$

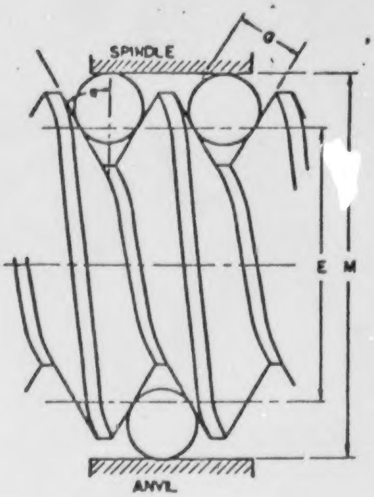
$C = .0007$

Alternate Problem - solve for minimum material condition tolerance (given as .013)



Wire Site and Thread Gages

Calibration of male thread gages usually requires a greater degree of precision than can be obtained using thread micrometers, calipers or even projection comparators (shadow graphs). The customary method involves a flat anvil caliper measurement over three hardened cylindrical wires of specific size to fit the particular thread. The wires, made of hardened tool steel and round, straight and accurate in size to 0.00002 inch, are placed in the thread, two on one side and one on the other as shown below.



BEST-WIRE SIZES FOR THREADS

N = No. Thds. per Inch	$G = \frac{.57735}{N}$ Dia. Best Wire	N = No. Thds. per Inch	$G = \frac{.57735}{N}$ Dia. Best Wire	N = No. Thds. per Inch	$G = \frac{.57735}{N}$ Dia. Best Wire
80	.00722	28	.02062	11-1/2	.05020
72	.00802	26	.02221	11	.05249
64	.00902	24	.02406	10	.05774
56	.01031	22	.02624	9	.06415
50	.01155	20	.02887	8	.07217
48	.01203	19	.03039	7	.05248
44	.01312	18	.03207	6	.09622
40	.01443	16	.03608	5-1/2	.10497
36	.01604	14	.04124	5	.11547
32	.01804	13	.04441	4-1/2	.12830
30	.01924	12	.04811	4	.14434

The table indicates the best wire size for national form threads. As long as the wires are small enough to be tangent to the sides of the threads and large enough that a measurement can be made without striking the crest of the thread, any error in the thread angle will be included in the measurement.

The formula for computing the pitch diameter from the measurement made over wires (for American national form-60 degrees included angle) is:

$$E = M + \frac{0.86603}{N} - 3G$$

where E = pitch diameter

N = number of threads per inch

G = diameter of wires

M = measurement over the wires

Calculate the pitch diameter for a 24 thread per inch gage, if the measurement over the wires is 0.384 inch

$$E = 0.384 + 0.86603/24 - 3(0.02406) \text{ from table}$$

$$= 0.384 + 0.3608 - 0.7218$$

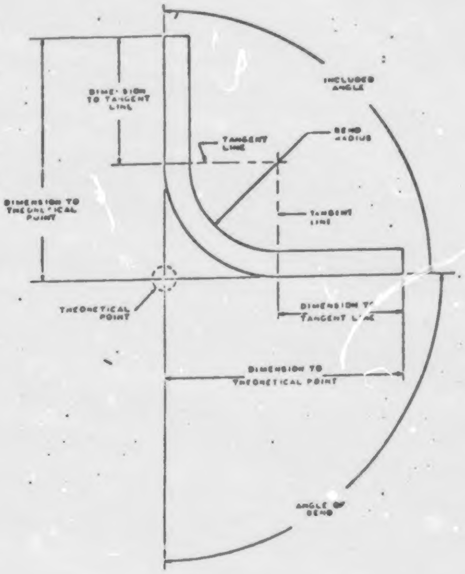
$$= 0.3479 \text{ inch}$$

pitch diameter = 0.3479 inch

Bend Allowance/Bend Deduction

When sheet metal is bent, the outer portion will stretch and the inner portion will compress. Typical drawings dimension the part to the theoretical tangent point, requiring an added material allowance to form the bend radius around the bend to the other edge. This length is called the bend allowance.

The inspector must reduce the dimensioned length to follow the curvature around the bend radius (always the inside of the material). The length reduction is called the bend deduction.


INCLUDED ANGLE
DIMENSION TO TANGENT LINE
TANGENT LINE
BEND RADIUS
DIMENSION TO THEORETICAL POINT
TANGENT LINE
THEORETICAL POINT
DIMENSION TO TANGENT LINE
DIMENSION TO THEORETICAL POINT
ANGLE OF BEND

The formula for calculating the bend deduction is:

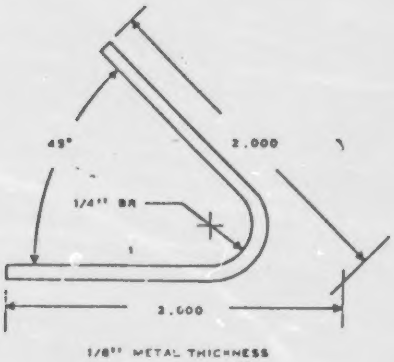
Bend Deduction = $2(R \cdot T) \cdot$ (Tangent of $\frac{1}{2}$ bend angle)-(bend allowance)

where R = radius, T = material thickness

The formula for calculating the bend allowance is:

Bend Allowance = Angle $\cdot$ $(0.01745R + 0.0078T)$

Calculate the Bend Deduction for:


45°
2.000
1/4" BR
2.000
1/8" METAL THICKNESS

Bend Deduction

$= 2\ (R \cdot T) \cdot \text{Tan}\ \frac{1}{2}\ \text{Bend Angle} - \text{Bend Allowance}$

$= 2\ (0.25 + 0.125) \cdot \text{Tan}\ 67^{\circ}\ 30' - 135^{\circ}\ (0.01745 \cdot 0.25 + 0.0078 \cdot 0.12)$

$= 2\ (0.375) \cdot 2.4142 - 135^{\circ}\ (0.0043625 + 0.000975)$

$= 0.75 \cdot 2.4142 - 135^{\circ}\ (0.0053375)$

$= 1.81065 - 0.72056$

$= 1.09009$

Bend Deduction = 1.090 inches

Thread Accuracy

The control of thread accuracy is important in industries where product performance depends on the ability of threads to withstand service and to maintain their adjustment over long periods of time. The National Bureau of Standards publishes Screw Thread Standards, Handbook H-28 for most standard threads, however, special thread sizes frequently require calculations of allowances and tolerances from unified formulas in which the pitch diameter tolerances are based on increments of the basic major (nominal) diameter, the pitch and the length of engagement.

The basic formula from which allowances on all diameters and tolerances on pitch diameters are derived is:

$$\text{Tolerance (allowance)} = C(0.0015\sqrt[3]{D} + 0.0015\sqrt{L_e} + 0.015\sqrt[3]{p^2}$$

C = Thread Class Tolerance Factor

Class	Factor C
1A	0.300
2A	0.300
3A	0.000

D = Basic Major Diameter

L_e = Length of Engagement

p = pitch

Increments in pitch-diameter tolerance formula[1]

(PD tolerance $= C(0.0015\sqrt[3]{D}+0.0015\sqrt{L_e}+0.015\sqrt[3]{p^2})$)

Diameter, D				Length of engagement, L_e															
D	$0.0015\sqrt[3]{D}$	D	$0.0015\sqrt[3]{D}$	Based on			L_e	$0.0015\times\sqrt{L_e}$	Based on			L_e	$0.0015\times\sqrt{L_e}$	Based on			L_e	$0.0015\times\sqrt{L_e}$
				1 D for sizes	9p for tpi	20p for tpi			1 D for sizes	9p for tpi	20p for tpi			1 D for sizes	9p for tpi	20p for tpi		
in.	in.	in.	in.				in.	in.				in.	in.				in.	in.
0.0600	0.000587	1.9375	0.001870	#0			0.0600	0.000367	7/16			0.4375	0.000992				3.1250	0.002652
.0625	.000595	2.0000	.001890				.0625	.000375		20		.4500	.001006	3 1/4			3.2500	.002704
.0730	.000627	2.0625	.001909	#1			.0730	.000405			44	.4545	.001011			6	3.3333	.002739
.0860	.000662	2.1250	.001928				.0781	.000419	1/2	18	40	.5000	.001061				3.3750	.002756
.0938	.000682	2.1875	.001947	#2			.0860	.000440			36	.5556	.001118	3 1/2			3.5000	.002806
.0990	.000694	2.2500	.001966				.0933	.000459	9/16	16		.5625	.001125				3.6250	.002856
.1120	.000723	2.3125	.001984	#3			.0990	.000472	5/8		32	.6250	.001186	3 3/4			3.7500	.002905
.1250	.000750	2.3750	.002001				.1094	.000496		14		.6429	.001203				3.8750	.002953
.1380	.000775	2.4375	.002019	#4			.1120	.000502				.6875	.001244	4		5	4.0000	.003000
.1640	.000821	2.5000	.002036		80		.1125	.000503			28	.7143	.001268				4.1250	.003047
.1875	.000858	2.6250	.002069	#5	72		.1250	.000530			27	.7407	.001291	4 1/4			4.2500	.003092
.1900	.000862	2.7500	.002102	#6			.1380	.000557	3/4	12		.7500	.001299				4.3750	.003137
.2160	.000900	2.8750	.002133		64		.1406	.000562				.8125	.001352	4 1/2			4.5000	.003182
.2500	.000945	3.0000	.002163				.1563	.000593			24	.8333	.001369				4.6250	.003226
.3125	.001018	3.1250	.002193		56		.1607	.000601	7/8			.8750	.001403	4 3/4			4.7500	.003269
.3750	.001082	3.2500	.002222	#8			.1640	.000607		10		.9000	.001423				4.8750	.003312
.4375	.001139	3.3750	.002250				.1719	.000622				.9375	.001452	5		4	5.0000	.003354
.5000	.001191	3.5000	.002277		48		.1875	.000650	1	9	20	1.0000	.001500				5.1250	.003396
.5625	.001238	3.6250	.002304	#10			.1900	.000654				1.0625	.001546	5 1/4			5.2500	.003437
.6250	.001282	3.7500	.002330				.2031	.000676			18	1.1111	.001581				5.3750	.003478
.6875	.001324	3.8750	.002356		44		.2045	.000678	1 1/8	8		1.1250	.001591	5 1/2			5.5000	.003518
.7500	.001363	4.0000	.002381	#12			.2160	.000697				1.1875	.001635				5.6250	.003558
.8125	.001400	4.2500	.002430				.2188	.000702	1 1/4		16	1.2500	.001677	5 3/4			5.7500	.003597
.8750	.001435	4.5000	.002476		40		.2250	.000712				1.3125	.001718				5.8750	.003636
.9375	.001468	4.7500	.002521				.2344	.000726	1 3/8			1.3750	.001759	6			6.0000	.003674
1.0000	.001500	5.0000	.002565	1/4	36	80	.2500	.000750			14	1.4286	.001793				6.5000	.003824
1.0625	.001531	5.2500	.002607				.2656	.000773				1.4375	.001798				7.0000	.003969
1.1250	.001560	5.5000	.002648			72	.2778	.000791	1 1/2	6		1.5000	.001837				7.5000	.004108
1.1875	.001588	5.7500	.002687		32		.2812	.000795	1 5/8			1.6250	.001912				8.0000	.004243
1.2500	.001616	6.0000	.002726				.2969	.000817			12	1.6667	.001936				8.5000	.004373
1.3125	.001642	7.0000	.002869	5/16		64	.3125	.000839	1 3/4			1.7500	.001984				9.0000	.004500
1.3750	.001668	8.0000	.003000		28		.3214	.000850	1 7/8			1.8750	.002054				9.5000	.004623
1.4375	.001693	9.0000	.003120				.3281	.000859	2	4 1/2	10	2.0000	.002121				10.0000	.004743
1.5000	.001717	10.0000	.003232		27	60	.3333	.000866	2 1/8			2.1250	.002187				10.5000	.004861
1.5625	.001741	12.0000	.003434				.3438	.000880	2 1/4	4		2.2500	.002250				11.0000	.004975
1.6250	.001764	14.0000	.003615			56	.3571	.000896				2.3750	.002312				11.5000	.005087
1.6875	.001786	16.0000	.003780				.3594	.000899	2 1/2		8	2.5000	.002372				12.0000	.005196
1.7500	.001808	18.0000	.003931	3/8	24		.3750	.000919				2.6250	.002430					
1.8125	.001829	20.0000	.004072				.3906	.000937	2 3/4			2.7500	.002487					
1.8750	.001850	24.0000	.004327				.4063	.000956				2.8750	.002543					
						48	.4167	.000968	3			3.0000	.002598					
							.4219	.000974										

Pitch, p											
Threads per inch	$0.015\sqrt[3]{p^2}$	Threads per inch	$0.015\sqrt[3]{p^2}$	Threads per inch	$0.015\sqrt[3]{p^2}$	Threads per inch	$0.015\sqrt[3]{p^2}$	Threads per inch	$0.015\sqrt[3]{p^2}$	Threads per inch	$0.015\sqrt[3]{p^2}$
	in.		in.		in.		in.		in.		in.
80	0.000808	50	0.001105	36	0.001376	27	0.001667	18	0.002184	11 1/2	0.002944
72	.000867	48	.001136	34	.001429	26	.001709	16	.002362	11	.003033
64	.000938	44	.001204	32	.001488	24	.001803	14	.002582	10	.003232
60	.000979	42	.001241	30	.001554	22	.001910	13	.002713	9	.003467
56	.001025	40	.001282	28	.001627	20	.002036	12	.002862	8	.003750

Threads per inch	$0.015\sqrt[3]{p^2}$
	in.
7	0.004099
6	.004543
5 1/2	.004814
5	.005130
4 1/2	.005503
4	.005953

Using the table provided, calculate the allowance for a ½-28 UNEF-2A thread. The basic size is 0.500 (½) inch diameter, with 28 threads per inch, of the Unified Extra Fine Series (UNEF) and is a class 2 external thread (2A).

$$\text{Allowance} = C(0.0015^3\ D\ +\ 0.0015\ \ L\ \ +\ 0.015^3\ p^2\ )$$

$$= 0.300(0.001191 + 0.001061 + 0.001627)$$

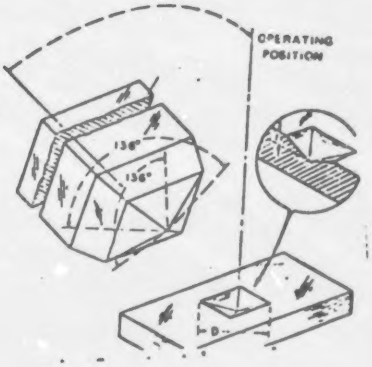
$$= 0.300(0.003879)$$

$$= 0.0011$$

Microhardness

Small precision parts, surface layers and cutting tool gages are only a few of the many situations in which microhardness testing is utilized to determine the hardness of the material used in fabrication of products.

The term microhardness usually refers to static indentation tests made with loads (weights) up to 1000 grams. An indentation in the material is obtained by applying the load, without friction or impact, to an indenter which is either the 136° diamond pyramid (Vickers) or the Knoop dismond indenter.



Material Hardness

The extremely shallow indentation is microscopically measured to the nearest millimeter and converted to the diamond pyramid hardness number (DPH), which is sometimes specified as the material hardness requirement or can be converted to other hardness requirement specifications using a table similar to the one below.

Determine the approximate tensile strength for the following material using the formula:

$$DPH = \frac{2P \sin\theta/2}{d^2}$$

p = applied load (KG) = 10 KG

d = mean diagonal of indentation (mm) = .1393 mm

θ = angle between opposite faces of the diamond = 136^0

$$DPH = \frac{(2)\ (10 \sin 136\ /2)}{(.1393\ mm)^2}$$

$$= \frac{(2)\ (10)\ (\sin 66\ )}{(.1393\ mm)^2}$$

$$= \frac{(20)\ (.9135)}{.0194\ mm^2}$$

$$= \frac{18.27}{.0194}\ mm^2$$

= 941.75 DPH

Per table, tensile strength is less than 301 KSI.

CONVERSION CHART

HARDENED STEEL AND HARD ALLOYS

ROCKWELL SCALE							DPH 10 Kg.	KHN* 500 Gr. & Over	BHN 3000 Kg.	Tensile Strength KSI Approx. Only
C	A	D	15N	30N	45N	G				
80	92.0	86.5	96.5	92.0	87.0	–	1865	–	–	
79	91.5	85.5	–	91.5	86.5	–	1787	–	–	
78	91.0	84.5	96.0	91.0	85.5	–	1710	–	–	
77	90.5	84.0	–	90.5	84.5	–	1633	–	–	
76	90.0	83.0	95.5	90.0	83.5	–	1556	–	–	
75	89.5	82.5	–	89.0	82.5	–	1478	–	–	
74	89.0	81.5	95.0	88.5	81.5	–	1400	–	–	
73	88.5	81.0	–	88.0	80.5	–	1323	–	–	
72	88.0	80.0	94.5	87.0	79.5	–	1245	–	–	
71	87.0	79.5	–	86.5	78.5	–	1160	–	–	
70	86.5	78.5	94.0	86.0	77.5	–	1076	972	–	
69	86.0	78.0	93.5	85.0	76.5	–	1004	946	–	
68	85.5	77.0	–	84.5	75.5	–	942	920	–	
67	85.0	76.0	93.0	83.5	74.5	–	894	895	–	
66	84.5	75.5	92.5	83.0	73.0	–	854	870	–	
65	84.0	74.5	92.0	82.0	72.0	–	820	846	–	
64	83.5	74.0	–	81.0	71.0	–	789	822	–	
63	83.0	73.0	91.5	80.0	70.0	–	763	799	–	
62	82.5	72.5	91.0	79.0	69.0	–	739	776	–	–
61	81.5	71.5	90.5	78.5	67.5	–	716	754	–	–
60	81.0	71.0	90.0	77.5	66.5	–	695	732	614	–
59	80.5	70.0	89.5	76.5	65.5	–	675	710	600	–
58	80.0	69.0	–	75.5	64.0	–	655	690	587	–
57	79.5	68.5	89.0	75.0	63.0	–	636	670	573	–
56	79.0	67.5	88.5	74.0	62.0	–	617	650	560	–
55	78.5	67.0	88.0	73.0	61.0	–	598	630	547	301
54	78.0	66.0	87.5	72.0	59.5	–	580	612	534	291
53	77.5	65.5	87.0	71.0	58.5	–	562	594	522	282
52	77.0	64.5	86.5	70.5	57.5	–	545	576	509	273
51	76.5	64.0	86.0	69.5	56.0	–	528	558	496	264
50	76.0	63.0	85.5	68.5	55.0	–	513	542	484	255
49	75.5	62.0	85.0	67.5	54.0	–	498	526	472	246
48	74.5	61.5	84.5	66.5	52.5	–	485	510	460	237
47	74.0	60.5	84.0	66.0	51.5	–	471	495	448	229
46	73.5	60.0	83.5	65.0	50.0	–	458	480	437	221
45	73.0	59.0	83.0	64.0	49.0	–	446	466	426	214
44	72.5	58.5	82.5	63.0	48.0	–	435	452	415	207
43	72.0	57.5	82.0	62.0	46.5	–	424	438	404	200
42	71.5	57.0	81.5	61.5	45.5	–	413	426	393	194
41	71.0	56.0	81.0	60.5	44.5	–	403	414	382	188
40	70.5	55.5	80.5	59.5	43.0	–	393	402	372	182
39	70.0	54.5	80.0	58.5	42.0	–	383	391	362	177
38	69.5	54.0	79.5	57.5	41.0	–	373	380	352	171
37	69.0	53.0	79.0	56.5	39.5	–	363	370	342	166
36	68.5	52.5	78.5	56.0	38.5	–	353	360	332	162
35	68.0	51.5	78.0	55.0	37.0	–	343	351	322	157
34	67.5	50.5	77.0	54.0	36.0	–	334	342	313	153
33	67.0	50.0	76.5	53.0	35.0	–	325	334	305	148
32	66.5	49.0	76.0	52.0	33.5	–	317	326	297	144
31	66.0	48.5	75.5	51.5	32.5	–	309	318	290	140
30	65.5	47.5	75.0	50.5	31.5	92.0	301	311	283	136
29	65.0	47.0	74.5	49.5	30.0	91.0	293	304	276	132
28	64.5	46.0	74.0	48.5	29.0	90.0	285	297	270	129
27	64.0	45.5	73.5	47.5	28.0	89.0	278	290	265	126
26	63.5	44.5	72.5	47.0	26.5	88.0	271	284	260	123
25	63.0	44.0	72.0	46.0	25.5	87.0	264	278	255	120
24	62.5	43.0	71.5	45.0	24.0	86.0	257	272	250	117
23	62.0	42.5	71.0	44.0	23.0	84.5	251	266	245	115
22	61.5	41.5	70.5	43.0	22.0	83.5	246	261	240	112
21	61.0	41.0	70.0	42.5	20.5	82.5	241	256	235	110
20	60.5	40.0	69.5	41.5	19.5	81.0	236	251	230	108

*Knoop Hardness Conversion - The values of the Knoop Hardness Number are approximate only, as they were determined on a limited number of tests and samples. These values are only for loads of 500 grams or heavier.

NOTE: Although conversion tables dealing with hardness can only be approximate and never mathematically exact, it is of considerable value to be able to compare different hardness scales in a general way.

This table is based on the assumption that the metal tested is homogeneous to a depth several times as great as the depth of the indentation because different loads and different shapes of penetrators would, in metal not homogeneous, penetrate – or at least meet the resistance of – metal of varying hardness, depending upon the depth of the indentation. Hence, no recorded hardness value would be valid to an extent that could be confirmed by another person unless shape of penetrator and actual load applied are both specified.

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Standardization of Parts

When Eli Whitney accepted a contract to supply the U. S. Army with several thousand muskets many years ago, a skilled gunsmith could manufacture fewer than ten muskets per year. His "hair-brained" scheme was to manufacture parts "so much alike that the firing pin from any one musket would fit any other musket, so that the barrel from one could be removed and another barrel could be fitted in its place."

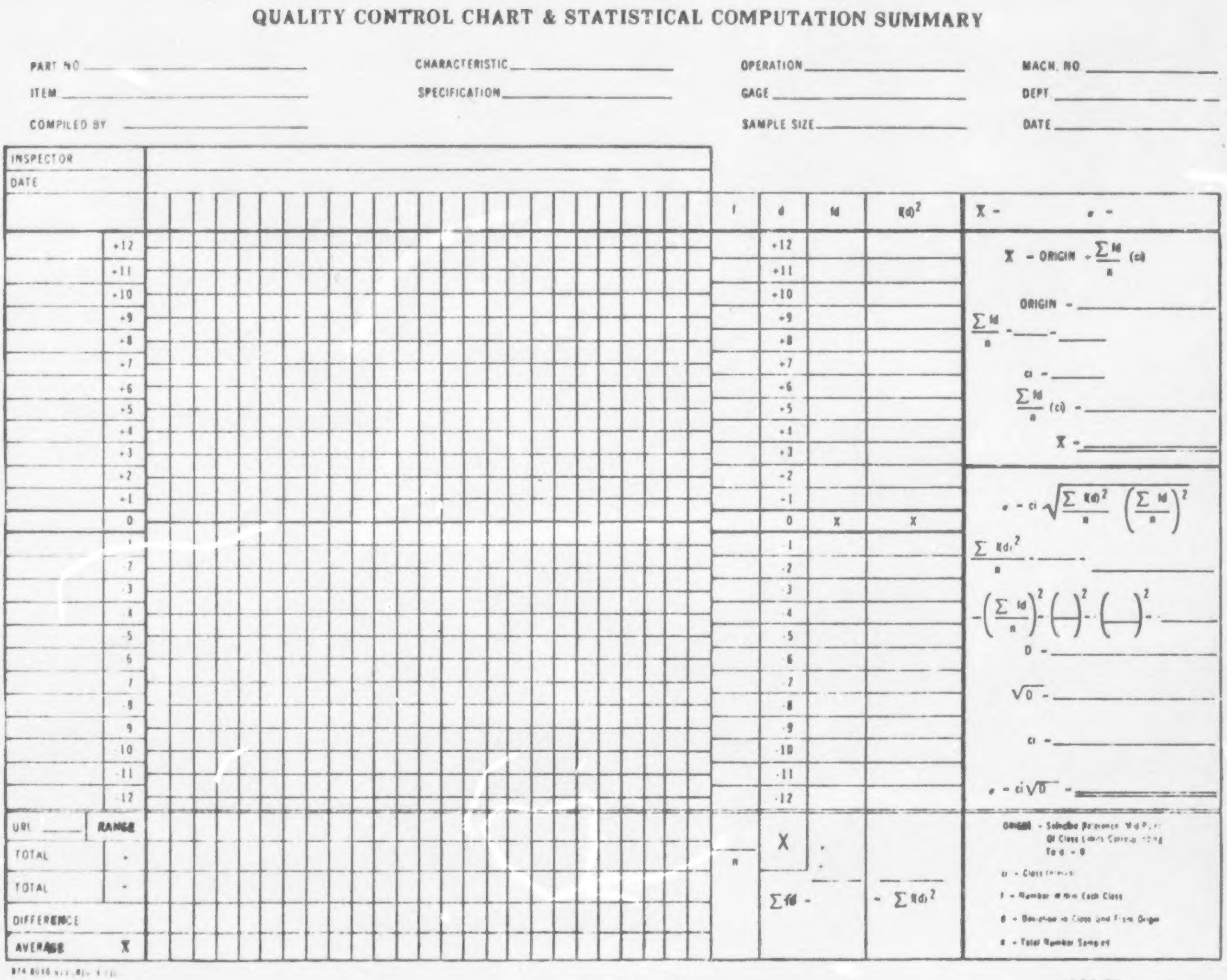
His successful application of the "interchangeable parts" concepts introduced the need for standardization of parts as well as a control over variability inherent in any process. We accept this concept today as "tolerancing."

Control of process considers not only the acceptability of individual parts produced by a process, but also the inherent variation in the process itself. In order to maintain a process within predefined limits, thereby assuring that process variability will not adversely affect the end product, mathematical (or more precisely statistical) techniques have been developed to simplify and standardize predictive control of the process.

By portraying this data in a chart or a graph, the operator and other interested parties can determine at a glance the status of the process and make adjustments as necessary. Not only are these techniques beneficial for preventive maintenance, tool wear and improved product quality, perhaps the most significant benefit is the constant awareness provided to the operator and inspector that the product is likely well within requirements or that it is potentially marginal, based upon the condition or status of the process.

One of the tools most frequently used to monitor a process statistically is the $\bar{X}$-R chart or Shewart Control Chart developed at Bell Telephone Laboratories in the 1940's.

The following chart is a standard computation summary:



QUALITY CONTROL CHART & STATISTICAL COMPUTATION SUMMARY

PART NO ____________ CHARACTERISTIC ____________ OPERATION ____________ MACH. NO ____________

ITEM ____________ SPECIFICATION ____________ GAGE ____________ DEPT ____________

COMPILED BY ____________ SAMPLE SIZE ____________ DATE ____________

INSPECTOR

DATE

		f	d	fd	f(d)^2
	+12		+12		
	+11		+11		
	+10		+10		
	+9		+9		
	+8		+8		
	+7		+7		
	+6		+6		
	+5		+5		
	+4		+4		
	+3		+3		
	+2		+2		
	+1		+1		
	0		0	X	X
	-1		-1		
	-2		-2		
	-3		-3		
	-4		-4		
	-5		-5		
	-6		-6		
	-7		-7		
	-8		-8		
	-9		-9		
	-10		-10		
	-11		-11		
	-12		-12		
URL ____	RANGE	n	X	$\sum fd$ =	= $\sum f(d)^2$
TOTAL	+				
TOTAL	-				
DIFFERENCE					
AVERAGE	$\bar{X}$				

$\bar{X}$ = σ =

$$\bar{X} = \text{ORIGIN} + \frac{\sum fd}{n}(ci)$$

ORIGIN = ____________

$\frac{\sum fd}{n}$ = ______ = ______

ci = ______

$\frac{\sum fd}{n}(ci)$ = ____________

$\bar{X}$ = ____________

$$\sigma = ci\sqrt{\frac{\sum f(d)^2}{n} - \left(\frac{\sum fd}{n}\right)^2}$$

$\frac{\sum f(d)^2}{n}$ = ______ = ____________

$-\left(\frac{\sum fd}{n}\right)^2 = \left(\quad\right)^2 = \left(\quad\right)^2 =$ ______

D = ____________

$\sqrt{D}$ = ____________

ci = ____________

$\sigma = ci\sqrt{D}$ = ____________

ORIGIN = Selected Reference Mid Point Of Class Limits Corresponding To d = 0

ci = Class Interval

f = Number Within Each Class

d = Deviation in Class Unit From Origin

n = Total Number Sampled

In constructing an $\bar{X}$-R chart, one needs only to understand the basic concepts of calculating the average ($\bar{X}$) and range (R) of a sequential set of measurements. The statistical formulas for calculate process control limits can be eliminated by utilizing tables which eliminate the need for extensive computation.

The average is calculated by adding together all the measurements taken and dividing this total by the number of measurements.

$$\text{Average } (\bar{X}) = \frac{\text{sum of measurements}}{\text{number of measurements}}$$

The simplest measure of variation of a process is the range, which is the difference between the largest and smallest measurements.

Range (R) = largest measurement - smallest measurement

If a lathe operation is set up to turn shafts to a diameter of 1.745±.005 inch and five shafts were measured to determine the process condition, calculate the average and range for these measurements:

1.245, 1.247, 1.244, 1.246, 1.243.

$$\bar{X} = \begin{array}{r} 1.245 \\ 1.247 \\ 1.244 \\ 1.246 \\ \underline{1.243} \\ 6.225 \end{array} \qquad \begin{array}{r} 1.245 \\ 5\,\overline{)6.225} \\ \underline{5} \\ 1\,2 \\ \underline{1\,0} \\ 22 \\ \underline{20} \\ 25 \\ \underline{25} \\ 0 \end{array} \qquad R = \begin{array}{r} 1.247 \\ \underline{-1.243} \\ .004 \end{array}$$

Using these values, we can predict the likelihood that parts produced will continue to meet specification requirements. An engineer factor A_2 derived statistically and proven in practice is utilized to establish limits on the process, within which we can expect the process to produce parts, considering it inherent variation and the number of measurements taken.

Calculate the control limits using the following formula, if $A_2 = .577$

$$\text{Control limits} = \bar{X} \pm A_2 R = 1.245 \pm .577(.004) =$$

$$1.245 \pm .0023 = \begin{array}{l} 1.2473 \\ 1.2427 \end{array}$$

Arc Time (Welding)

Acceptable welds depend upon many factors all done correctly. Sometimes insignificant related issues can be as significant as those parameters necessary to actually produce the weld. Most welding operations are performed to pre-calculated time standards or "guesstimates" designed to estimate how much actual time a required operation will take. Arc time is one factor which not only is significant in producing a quality weld, but also an essential factor which directly affects the welders is time standard performance. A welder wants to know ahead of time how much time the job is supposed to take, and wants to be compensated for beating that time or "making out" as it is sometimes called.

Arc time can be measured by an arc clock or it can be calculated mathematically from a simple formula that takes into account fillet area (AF), wire area (WA), wire speed (WS), and the efficiency of the process (PE) which is determined from process history, as follows:

$$\frac{(FA)\ \text{sq.in.}}{(WA)\ \text{sq.in.}\ \times\ (WS)\ \text{in/min.}\ \times\ (PE)} = \text{Arc min/in}$$

Pre-calculating a fillet factor (FF) based upon the electrode wire used, the commonly used fillets and the applicable process can save calculation time by realizing that

$$(FA)/(WA) \times (PE)\text{min} = (FF)\text{min}$$

The only other factor to be considered for gas-metal-arc welding with wires of various size and the gap allowance (in minutes) for various gap sizes and fillet sizes. The gap allowance is established by determining the change in area for various gap conditions. Calculating arc time can be determined by

$$\frac{(FF) \times (GA)\text{min}}{(WS)\text{in}} = \text{Arc min/in}$$

The pre-calculated FF and GA are given by the following tables.

Save time by precalculating fillet factors and gap allowances

Fillet	Wire size, in.				Gap, thousandths of an inch							
	0.030	0.035	0.045	1/16	0.000	0.010	0.020	0.030	0.041	0.050	0.060	0.070
	Factor				Gap allowance							
1/8 in.	11.192	8.223	5.034	2.665	1.000	1.171	1.379	1.575	1.914	2.253	2.593	2.932
3/16	25.183	18.503	11.326	5.997	1.000	1.112	1.240	1.364	1.550	1.737	1.924	2.110
1/4	44.771	32.895	20.135	10.662	1.000	1.083	1.175	1.263	1.388	1.513	1.638	1.763
5/16	69.954	51.398	31.461	16.659	1.000	1.066	1.138	1.207	1.300	1.393	1.486	1.578
3/8	100.734	74.013	45.304	23.989	1.000	1.055	1.113	1.171	1.244	1.317	1.390	1.463

Calculate the arc time for a 3/16" fillet with a gap of 0.010" using a gas-metal-arc process with a 0.045" diameter wire at a wire speed of 400 in/min.

$$\frac{(FF) \times (GA)\ \text{min}}{(WS)\ \text{in}} = \text{Arc min/in.}$$

$$\frac{11.326 \times 1.112}{400} = 0.0315\ \text{arc min/in.}$$

AUTOMOTIVE APPLICATION PROBLEMS

There are 10 total application problems dealing with 7 mathematical topics. These 7 topics include percentages, formula manipulation, geometry, direct and indirect variation, systems of equations, trigonometry, and quadratic equations. The breakdown is as follows:

Problem I	Percentages (discount)
Problems II - IV	Formula Manipulation
Problems V - VI	Geometry
Problem VII	Variation
Problem VIII	Systems of Equations
Problem IX	Trigonometry
Problem X	Quadratic Equations

I. **PROBLEM**:

A mechanic purchases 7½ feet of 11/32 - inch air-conditioning hose at $2.92 per foot. A discount of 40% is given. What does the mechanic pay for the hose to the nearest cent?

SOLUTION:

First, we find out how much the mechanic pays for the hose before the discount. We do this by multiplying

$$(7\tfrac{1}{2}\text{ feet})(\$2.92/\text{foot}) = \$21.90$$

We then take a 40% discount on $21.90.

$$(\$21.90)(.40) = \$8.76 \text{ (discount)}$$

$$\$21.90 - \$8.76 = \boxed{\$13.14}$$

The mechanic pays $13.14 for the hose.

II. PPOBLEM:

S.A.E. Horsepower is a method of estimating the horsepower of an engine. It is used as a rating to determine the amount to be paid for taxation. The formula for S.A.E. Horsepower is:

$$hp = \frac{D^2N}{2.5}$$

Where: hp is the horsepower
D is the cylinder bore (inches)
N is the number of cylinders

What is the cylinder bore of a 6 - cylinder engine with a horsepower rating of 21.6?

SOLUTION:

We are given the number of cylinders and the rating and want to find the cylinder bore in this problem.

Since our formula is solved for horsepower rating, we need to manipulate it by solving for D, or the cylinder bore.

$$hp = \frac{D^2N}{2.5}$$

$$(2.5)(hp) = D^2N$$

$$\frac{(2.5)(hp)}{N} = D^2$$

$$\sqrt{\frac{(2.5)(hp)}{N}} = D$$

Substituting our known values, hp = 21.6 and N = 6, we get:

$$D = \sqrt{\frac{(2.5)(21.6)}{6}}$$

$$= \sqrt{9} \qquad = \boxed{3 \text{ inches}}$$

So, the cylinder bore is 3 inches.

III. PROBLEM:

The capacity, in gallons, of cylindrical tanks can be found by using the formula:

$$C = \frac{0.7854\ D^2L}{231\ \text{cu. in. per gal.}}$$

Where: C is the capacity (in gallons)

D is the diameter of the tank (in inches)

L is the length of the tank (in inches)

231 cubic inches = 1 gallon

It is necessary to get a 50-gallon tank in a space 3 feet long. What size of diameter is needed? Express answer to nearest tenth.

SOLUTION:

We are given the capacity and length of the tank and want to find the diameter. Since our formula is solved for C (capacity), we must manipulate it by solving for D (diameter), our unknown variable.

$$C = \frac{0.7854\ D^2L}{231\ \text{cu. in. per gal}}$$

$$(231)(C) = 0.7854\ D^2L$$

$$\frac{(231)(C)}{(0.7854)(L)} = D^2$$

$$\sqrt{\frac{(231)(C)}{(0.7854)(L)}} = D$$

Substituting our known values, C = 50 and L = 36 (inches), we get:

$$D = \sqrt{\frac{(231)(50)}{(0.7854)(36)}}$$

$$= \sqrt{408.5}$$

$$= \boxed{20.2 \text{ inches}}$$

So, the diameter of the tank must be 20.2 inches.

IV. When a gas undergoes an adiabatic (constant-heat) process, the relationship between final and initial temperatures and pressures is as follows:

$$\frac{T_2}{T_1} = \frac{P_1}{P_2}^{\frac{1-\gamma}{\gamma}}$$

Where T_1 & T_2 are the initial and final temperatures respectively

P_1 & P_2 are the initial and final pressures respectively

γ is the ratio of specific heats at constant pressure and constant volume.

PROBLEM:

In a certain engine, the temperature and pressure at the beginning of the compression stroke are $T_1 = 560°F$ (absolute) and $P_1 = 15$ psi (pounds per square inch); at the end of the stroke the pressure is $P_2 = 665$ psi. If $\gamma = 1.40$, find the ending temperature, T_2.

SOLUTION:

In this problem, we need to find T_2 with respect to our given formula. We must first manipulate the given formula and solve for T_2.

$$\frac{T_2}{T_1} = \left(\frac{P_1}{P_2}\right)^{\frac{1-\gamma}{\gamma}}$$

$$T_2 = T_1\left[\left(\frac{P_1}{P_2}\right)\right]^{\frac{1-\gamma}{\gamma}}$$

Substituting our known values: $T_1 = 560$ F, $P_1 = 15$ psi, $P_2 = 665$ psi, and $\gamma = 1.40$

We get:

$$\begin{aligned} T_2 &= 560^\circ \left[\left(\frac{15 \text{ psi}}{665 \text{ psi}}\right)^{\frac{1-1.4}{1.4}}\right] \\ &\simeq 560^\circ \left[(.0225)\right]^{-.2857} \\ &\simeq 560^\circ \left[2.956\right] \\ &\simeq \boxed{1660^\circ\text{F (absolute)}} \end{aligned}$$

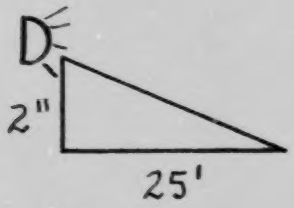
So, our ending temperature is ≃ (approximately) 1660°F abs.

V. PROBLEM:

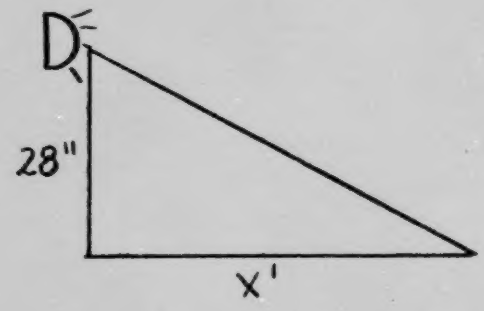
The state automobile inspection permits headlights set at an angle. The drop in the light beam cannot be greater than 2 inches for each 25 feet measured horizontally. The headlight is 28 inches above the ground. How far ahead of the car is the roadway lighted?

SOLUTION:

We can set up similar triangles for this problem.



State Automobile
Inspection



Our Problem
X = roadway distance

Since we know from geometry that ratios of corresponding sides of similar triangles are equivilent, we can set up a proportion.

$$\frac{2''}{25'} = \frac{28''}{X'}$$

Solving this proportion for X, we get:

$$2''(X') = (25')(28'')$$

$$X' = \frac{(25')(28'')}{2''}$$

$$= (25')(14)$$

$$= \boxed{350'}$$

So, 350 feet of roadway would be lighted if the headlight was 28 inches above the ground.

VI. **PROBLEM:**

A mechanic wishes to pour a concrete foundation for a storage garage for 85 cars. He wants each car to be allotted a space of 15 ft. by 10 ft. If the cost of the concrete is $\$9.70/\text{ft}^2$, what would be the total cost of the foundation?

SOLUTION:

We first need to find the total area of the foundation in square feet. We do this by multiplying the dimensions allotted for 1 car by the total number of cars, 85.

$$(15\text{ ft})(10\text{ ft})(85) = \text{Total Area}$$

$$12{,}750\ \text{ft}^2 = \text{Total Area}$$

We then multiply the number of square feet by the cost of 1 square foot to get the total cost.

$$(12{,}750\ \text{ft}^2)(\$9.70/\text{ft}^2) = \text{Total Cost}$$

$123,675 is the Total Cost of the foundation.

VII. PROBLEM:

The width of belt required in power transmission varies directly with the horsepower and inversely with the speed of the belt.

If a double belt traveling 600 ft/min will transmit 1 housepower for each inch of width, find the width of belt necessary to transmit 2 housepower with belt speed of 1200 ft/min.

SOLUTION:

Our equation of variation is:

$$W = \frac{KH}{R}$$

Where: W is the width of the belt (inches)
H is the horsepower
R is the speed of the belt (ft/min)
K is the constant of variation

To find K, our constant of variation, we substitute our known values of our initial condition: R = 600 ft/min, H = 1, W = 1 inch

$$W = \frac{KH}{R}$$

$$1 = \frac{K(1)}{600}$$

$$600 = K$$

With K = 600, we substitute our given values, H = 2 and R = 1200 ft/min, to find the unknown variable - W (width).

$$W = \frac{KH}{R}$$

$$W = \frac{(600)(2)}{1200}$$

$$= \frac{1200}{1200} = \boxed{1}$$

So, 1 inch would be the width of our belt.

VIII. PROBLEM:

A small, single-cylinder engine is operated on a test stand for 14 min. It was run first at 850 rpm, then at 1250 rpm during the test. A revolutions counter recorded a total of 15,500 revolutions during the test. How long was the engine operated at each speed?

SOLUTION:

Let X = Minutes operated at 850 rpm.

Let Y = Minutes operated at 1250 rpm.

We can set up 2 equations in terms of X and Y based on the information given.

Since the total number of minutes the machine was operated was 14, one equation is:

$$X + Y = 14$$

Since the total number of revolutions was 15,500 and the machine was operated for X minutes at 850 rpm and Y Minutes at 1250 rpm, we can identify the other equation:

$$850X + 1250Y = 15{,}500$$

So, we have 2 equations in 2 variables --

$$X + Y = 14$$

$$850X + 1250Y = 15{,}500$$

Solving, we solve the first equation for X:

$$X + Y = 14$$

$$X = 14 - Y$$

Substituting the 14 - Y for X in the second equation gives us:

$$850(14-Y) + 1250Y = 15{,}500$$

or

$$11900 - 850Y + 1250Y = 15{,}500$$

$$11900 + 400Y = 15{,}500$$

$$400Y = 3600$$

$$Y = 3600/400 = 9$$

So $\boxed{Y = 9 \text{ minutes}}$

Substituting Y = 9 in the first equation gives us:

$$X + 9 = 14$$

$$\boxed{X = 5 \text{ minutes}}$$

So, the machine was operated for 5 minutes at 850 rpm and 9 minutes at 1250 rpm.

IX. PROBLEM:

The motion of each piston in a reciprocating engine closely approximates simple harmonic motion. The speed of each piston is given by:

$$\omega Y_0 \sin(\omega t + \pi/2)$$

Where: ω is the angular speed of the crankshaft in radians per unit time.

Y_0 is the maximum displacement of the piston from the midpoint of its path.

Graph the speed vs. time, in seconds, of a piston of the Jaguar V-12 engine at 6,500 rpm. The stroke (or distance between extreme positions) of each piston is 2.76 in.

SOLUTION:

We will be graphing the function:

$$\omega Y_0 \sin(\omega t + \pi/2)$$

To find ω, the angular speed (rad/sec), we take the given rpm (6,500) and convert it to radians/sec.

$$6500 \, \frac{\text{revolutions}}{\text{min}} \times \frac{1 \text{ min}}{60 \text{ sec}} \times \frac{2\pi \text{ radians}}{1 \text{ revolution}}$$

$$= \frac{(6500)(2\pi)}{60} \, \frac{\text{radians}}{\text{sec}}$$

$$\approx 216\pi \, \frac{\text{rad}}{\text{sec}}$$ So, $\omega \approx 216\pi$

To find Y_0, the maximum displacement of the piston from its midpoint, we simply take half of the given extreme distance, 2.76 in.

So, $Y_0 = \frac{1}{2}(2.76) = 1.38$

Our function that we want to graph looks like:

$$(216\pi)(1.38) \sin (216\pi t + \pi/2)$$

or

$$940 \sin (216\pi t + \pi/2)$$

This function is of the form:

$$a \sin (bt + c)$$

Where: a is the amplitude (maximum height)

$\frac{2\pi}{b}$ is the period (range)

c is the phase shift

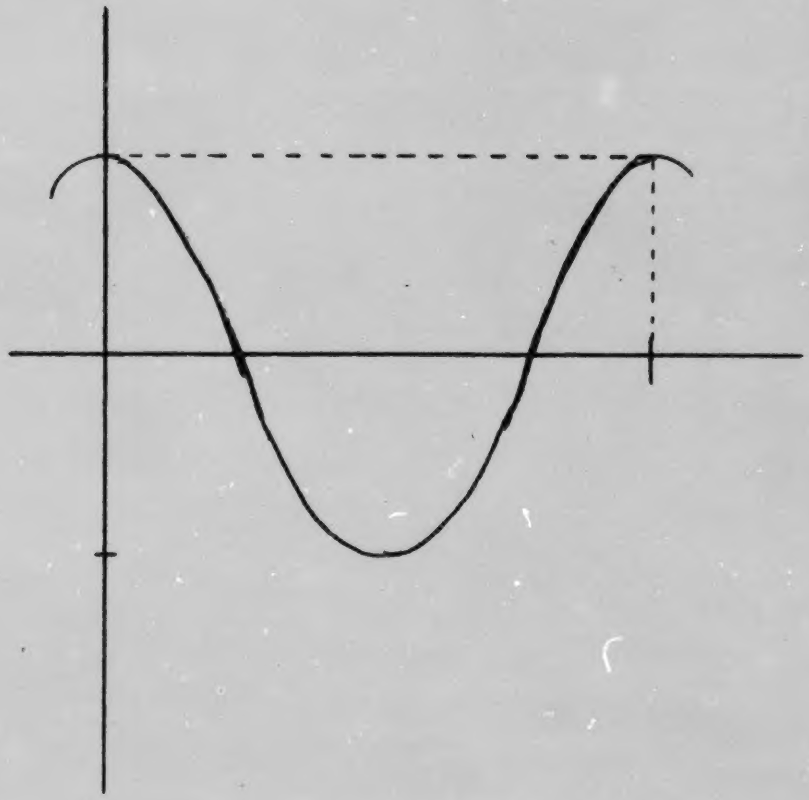
So, we want to graph the sin function with the following characteristics:

amplitude $= 940$

period $= \frac{2\pi}{216\pi} = \frac{1}{108}$

phase shift $= \pi/2$

Here is the graph:



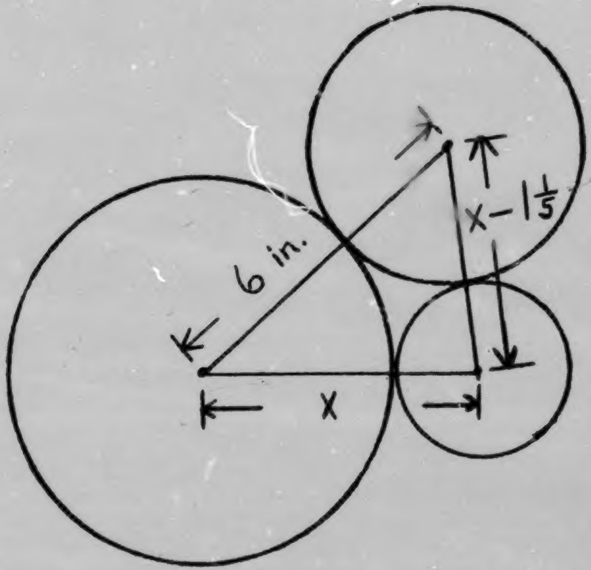
X. **PROBLEM**:

A certain automatic transmission can be locked by shifting 3 gears so that each one is externally tangent to each other and so that the lines between their centers form a right triangle. The center distance between the two largest gears is 6 in., and the distance between the two smallest is 1 1/5 in. less than the distance between the largest and smallest.

a) Find the distances between centers of these gears.

b) Find the radius of each gear.

SOLUTION:



We let x be the distance between centers
of the largest and smallest gears.

That means x - 1 1/5 would represent the distance
between centers of the two smallest gears.

Since 6, x, and x - 1 1/5 represent sides of a right triangle, we can use the Pythagorean Theorem to solve for x.

The Pythagorean Theorem states that the hypotemuse of a right triangle is equal to the square root of the sum of the squares of the other 2 sides.

i.e.

$$6 = \sqrt{x^2 + (x - \frac{6}{5})^2}$$

Simplifying this equation, we get:

$$36 = x^2 + (x - \frac{6}{5})^2$$

$$36 = x^2 + x^2 - \frac{12}{5}x + \frac{36}{25}$$

$$36 = 2x^2 - \frac{12}{5}x + \frac{36}{25}$$

$$2x^2 - \frac{12}{5}x - \frac{864}{25} = 0$$

$$50x^2 - 60x - 864 = 0$$

$$25x^2 - 30x - 432 = 0$$

Using the quadratic formula* with $a = 25$, $b = -30$, and $c = -432$, we get:

$$x = \frac{30 \pm \sqrt{(-30)^2 - 4(25)(-432)}}{2(25)}$$

$$= \frac{30 \pm \sqrt{900 + 4320}}{50}$$

$$= \frac{30 \pm \sqrt{44100}}{50}$$

$$= \frac{30 \pm 210}{50}$$

$$x = \frac{240}{50} \text{ and } \frac{-180}{50}$$

$$x = 4\frac{4}{5} \text{ and } -3\frac{3}{5}$$

Since x represents distance and distance cannot be negative, the solution $-3\frac{3}{5}$ is discarded. So, $x = 4\frac{4}{5}$ This implies that $x - 1\frac{1}{5} = 3\frac{3}{5}$

a) So, the distances between centers are:

$$\boxed{6 \text{ in}, \ 4\frac{4}{5} \text{ in}, \text{ and } 3\frac{3}{5} \text{ in.}}$$

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* $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

b) Let x be the radius of the smallest gear.

Let y be the radius of the middle gear.

Let z be the radius of the large gear.

Since the distance between the centers of the 2 largest gears is 6 in.,

then $y + z = 6$

Also, since the distance between the centers of the 2 smallest gears is $3\frac{3}{5}$ in.,

then $x + y = 3\frac{3}{5}$

likewise, $x + z = 4\frac{4}{5}$

This gives us three equations with three unknowns:

$$\begin{aligned} x + y \phantom{{}+z} &= 3\tfrac{3}{5} \\ y + z &= 6 \\ x + \phantom{y+{}} z &= 4\tfrac{4}{5} \end{aligned}$$

Solving this system using the substitution method, we get:

$$\boxed{\begin{aligned} x &= 1\tfrac{1}{5} \\ y &= 2\tfrac{2}{5} \\ z &= 3\tfrac{3}{5} \end{aligned}}$$

So, the radii of the three gears from smallest to largest are: $1\frac{1}{5}''$, $2\frac{2}{5}''$, $3\frac{3}{5}''$.

AVIATION - BASIC PRINCIPLES

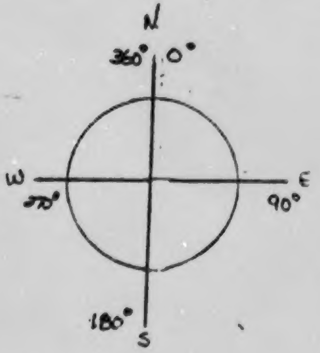
An Introduction

Cross country flights require pilots to have a knowledge of navigation, meteorology and cross-country flying techniques. Navigation is the science of conducting an airplane from one point on earth to another. Economy of time and distance are typically significant factors in flying due to fuel and fatigue considerations. Generally, the simplest methods possible are used by the pilot to check his position in the air. Dead-reckoning navigation knowledge, using the compass and calculation of courses and distances flown is quite frequently required, because even a few clouds, not to mention other more significant weather conditions can interfere with visual landmark navigational methods.

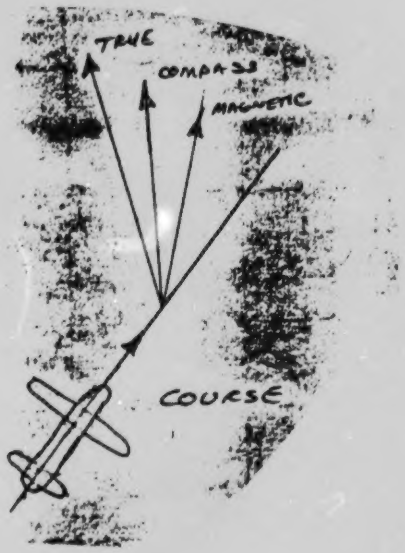
When using a compass to determine course direction, corrections must be made to the compass reading in order to determine the true heading. These corrections are required because of differences that exist, depending upon geographical location, between true north, magnetic north and compass north. Variation, the difference between true north and magnetic north, results from the magnetic North Pole being about 1000 miles south of the geographical North Pole. The difference between magnetic north and compass north is called deviation. Both variation and deviation errors are considered to be positive (+) if easterly and negative (-) if westerly. Since the direction or heading that an airplane is moving is called the course, there are three corresponding expressions used to describe the direction of travel, true course, magnetic course and compass north.

Problem:

If the compass (uncorrected) of an airplane indicates a course of 39° 30', the deviation is east 3° 15' and the variation is west 10° 45', determine the true course:


N
360°
0°
W
270°
E
90°
180°
S

Compass course		39° 30'
Deviation (East)	+	3° 15'
Variation (West)	-	10° 45'
		32° 0'


TRUE
COMPASS
MAGNETIC
COURSE

Further Applications

Airspeed indicators, a common type in older planes being a differential pressure type (pitot static), indicate the speed of an airplane relative to the air. Because angles of climb and dive increase or decrease airspeed, the airspeed indicator also indirectly registers changes in air speed to compensate for glide angle changes.

Errors due to velocity and direction of airflow patterns are minimized by selective placement of the pressure head where airflow disturbance is the least. Other errors, however, must be compensated for by the pilot (usually using pre-calculated charts) due to decreases in air pressure and density as altitude increases.

Since airspeed indicators are calibrated at standard barometric and temperature conditions (29.92 & 59° at sea level), airspeed indicators typically indicate two percent lower speed for each 1000 feet increase in altitude. A rough rule of thumb for airspeeds of 80 to 120 mph is to increase the indicated airspeed by 2 mph for each 1000 feet altitude. True airspeed will always be greater than indicated airspeed.

Problem:

Calculate the true airspeed, using both the 2% correction factor and the 2 mph rough rule of thumb for an airplane flying at 6000 feet with an indicated airspeed of 89 mph.

$$\text{True Airspeed} = \text{Indicated Airspeed} + (2\% \cdot \text{each } 1000 \text{ ft}) \cdot \text{Indicated Airspeed}$$

$$= 89 \text{ mph} + (0.02 \cdot 6)(89 \text{ mph})$$

$$= 89 + .12(89)$$

$$= 89 + 10.68$$

$$= 99.68 \text{ mph}$$

$$\text{True Airspeed} = \text{Indicated Airspeed} + (2 \text{ mph} \cdot \text{each } 1000 \text{ ft})$$

$$= 89 \text{ mph} + (2 \text{ mph} \cdot 6)$$

$$= 89 + 12$$

$$= 101 \text{ mph}$$

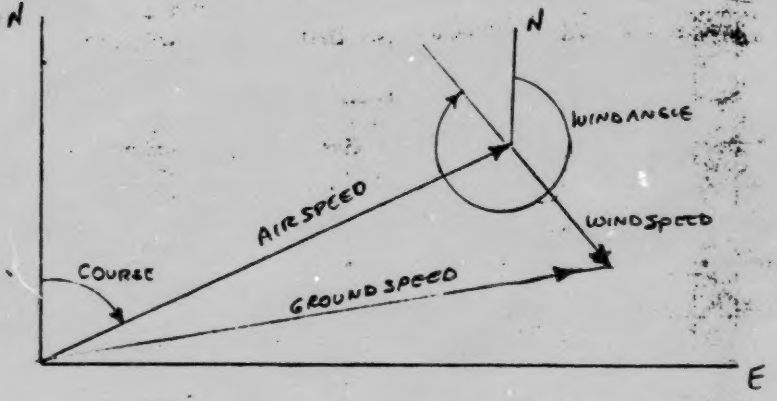
Airspeed

Airspeed, the speed of an airplane in still air, is determined from an airspeed indicator.

Ground position, the position on earth directly beneath the airplane, can be determined by calculating the groundspeed, the speed of the airplane with relation to the ground.

Wind speed and the wind angle, influence the airspeed in a vector relationship (add to or take from), and therefore affect the ground speed.

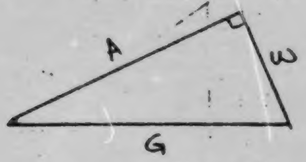
By solving trigonometric relationships of triangles, groundspeed is calculated from the known (or determined) airspeed, windspeed and their corresponding directional angles.


N
N
WIND ANGLE
AIRSPEED
WINDSPEED
COURSE
GROUNDSPEED
E

Problem:

Suppose you are flying at 200 mph on a course heading of 50° true. A crosswind of 45 mph from 310° is causing you to drift off course and the resulting groundspeed is not equivalent to your airspeed indicator reading solving the triangle relationship, determine your actual groundspeed.

Noting that the resulting triangle is a right (90°) triangle,



$$G^2 = A^2 + W^2$$

$$G = \sqrt{A^2 + W^2}$$

$$\text{Ground speed (G)} = \sqrt{\text{Airspeed (A)}^2 + \text{Windspeed (W)}^2}$$

$$= \sqrt{200^2 + 45^2}$$

$$= \sqrt{40{,}000 + 2025}$$

$$= \sqrt{42{,}025}$$

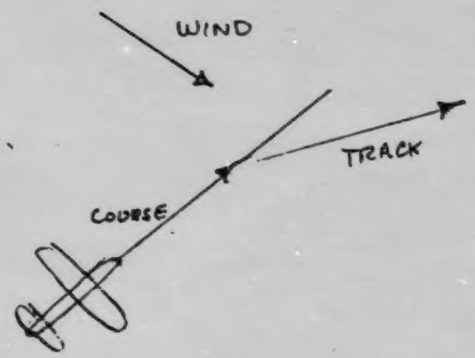
$$= 205 \text{ mph}$$

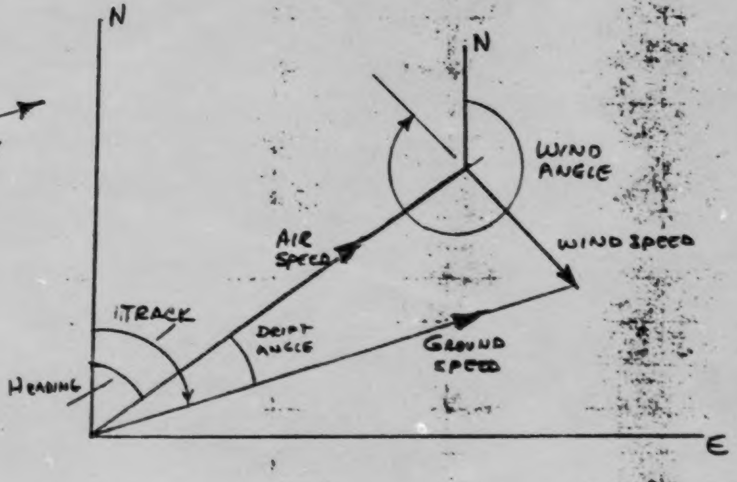
Note: Altitude correction factor information (problem 2) could be added to this problem, if desired.

Direction in Flight

The direction a plane is traveling in relation to the earth is called track and as in course expressions, there are three terms used to describe track, true track, magnetic track and compass track, corresponding to the course expression applicable. The difference between course and track is called drift with the term port used to describe drift to the left of the indicated course and starboard used to describe drift to the right.

Most generally, wind is the cause of drift and the indicated course must be compensated for these wind vectors to determine the actual course traveled. The drift angle (or wind correction angle) is the difference (positive) between the heading, (compass reading measured clockwise from north) and the track.

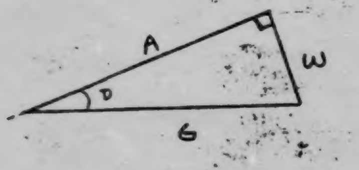

WIND
TRACK
COURSE


N
N
WIND ANGLE
AIR SPEED
WIND SPEED
TRACK
DRIFT ANGLE
GROUND SPEED
HEADING
E

Problem:

Suppose you are flying at 200 mph on a heading of 50° true. A crosswind of 45 mph from 310° is causing a starboard drift for which you must compensate in order to determine your actual course (track). Determine the track (actual true course).

In the resulting right triangle



Tangent or Drift Angle (D) = Windspeed (W)/Airspeed (A)

= 45/200

= 0.225

Drift Angle D = 12.68 or 12 41'

Track = 50 (heading) + 12 41' (drift Angle)

= 62 41'

Further Explanation of Airspeed and Groundspeed

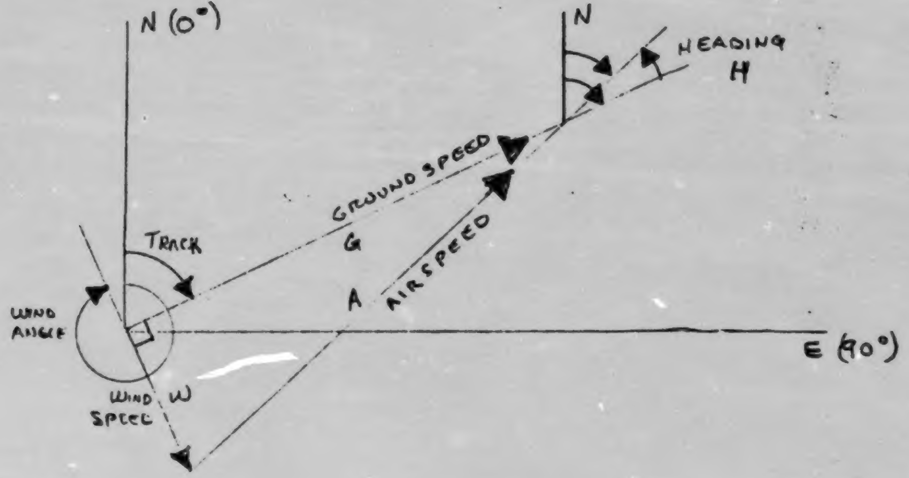
In the previous problems, trigonometric relationships were used to determine the error in airspeed and course direction caused by wind influences. Although it is theoretically possible that flying conditions could exist such that the influence of wind on an airplane would be negligible, usually, course corrections are necessary in order to arrive at a predetermined destination.

Although the techniques used to this point tell us where we should end up due to the influence of the wind on our flight path, we rarely want to end up at this point, some distance from our original destination.

Since we normally know where we want to end our flight, and the wind direction and velocity are known or can be obtained, we would probably want to determine an airspeed and heading such that the resulting groundspeed and track would assure that we would arrive at our intended destination.

Problem:

Suppose we have a flight plan from Dallas to Little Rock and need to be there in approximately 2 hours. A heading of 50^{c} at 250 mph would assure that we would reach our destination in the allotted time, however a 35 mph wind from 320 will alter our course as well as groundspeed. Calculate an airspeed and heading, taking into consideration the affects of the wind, which will result in the desired groundspeed and track previously determined.


N (0°)
N
HEADING
H
GROUNDSPEED
G
AIRSPEED
A
TRACK
WIND ANGLE
WIND SPEED
W
E (90°)

$$\text{Airspeed (A)} = \sqrt{\text{Groundspeed (G)}^2 + \text{windspeed (W)}^2}$$

$$= \sqrt{250^2 + 35^2}$$

$$= \sqrt{62500 + 1225}$$

$$= \sqrt{63725}$$

$$= 252.44 \text{ mph}$$

$$\text{Tangent Angle (H)} = \frac{\text{Windspeed (W)}}{\text{Groundspeed (G)}}$$

$$= 35/250$$

$$= 0.14$$

$$\text{Angle (H)} = 8^{0}$$

$$= \text{Track - Correction Angle}$$

$$\text{Heading} = 50^{0} - 8^{0}$$

$$= 42^{0}$$

Forces Acting on an Airplane

Two forces acting on an airplane in level flight are gravity which results from the weight of the plane (and its cargo) and lift, developed primarily by the wings, which must be balanced by the weight to maintain level flight. For take-offs and climbing, the lift, logically, must exceed the weight.

The most simplistic explanation of the lift force concerns only the airflow over the wing as compared to the air flowing beneath it. Since the air flowing over the wing profile has a greater distance to cover than the lower profile, the speed of the air above the wing is greater. The increased speed results in a pressure decrease perpendicular to the direction of airflow and a subsequent airflow increase below the wing, producing a lifting effect. Of course, the fuselage (body) tends to produce similar effects. Lift increases approx as the square of the velocity as long as the flight speed is maintained below the speed of sound (760 mph).

Factors that contribute to lift of an airplane also cause forces called Drag, which tend to retard the forward motion of the plane and must be overcome by propulsion. Objects moving through air create a resistance to the air, and a certain amount of drag is "induced" by the lift. Other types of drag, such as skin friction and turbulence contribute to the total drag. Designers try to design airplanes with much more lift than drag. Faster planes usually have lower lift to drag ratios.

Problem:

Assuming no drag or angle influences, determine the lift capability of a plane flying at a fixed altitude of wing relative to the oncoming air, at 113 mph and 72 mph (take off speed).

Lift = Velocity squared = V^2

Lift @ 113 mph = 113^2 = 12769 lbs.

Lift @ 72 mph = 72^2 = 5184 lbs.

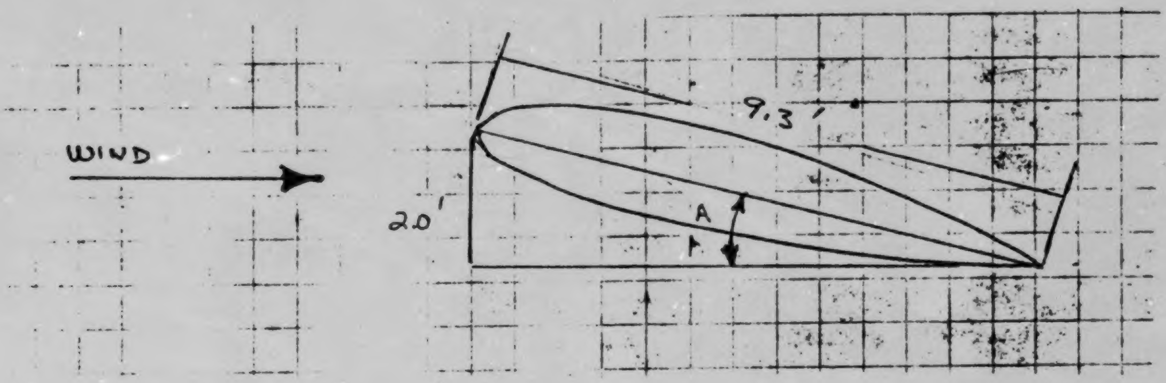
Lift varies approx linearly with the angle of attack. That is, the angle of the wing with respect to oncoming air. Because lift must always equal weight to maintain level flight, the angle of attack must be increased as speed decreases, such as for landings and takeoffs.

From a pilots view point, the angle of attack is the secret of flight, because no matter what else varies, such as speed or direction or altitude, the one factor which he really controls or changes is the angle of attack. The angle of attack does not relate to position relative to the horizon, but is measured specifically and exclusively with relation to the line of relative wind and flight path. For this reason, the true function of the air speed indicator is to measure the angle of attack. For every angle of attack, there is an appropriate lift and drag.

As the angle of attack is increased, air turbulence spreads over the wing and at some critical point, typically between 14° and 18°, the wing loses lift and the plane stalls.

Problem:

Given a wing profile with a chord length of 9.3 feet, as shown below, with a 30 mph headwind from 270° (west) creating a 2 foot lift at the leading edge of the wing of an airplane flying at 70 mph, calculate the angle of attack, A.



Sin A (Angle of Attack) = Opposite/Hypotenuse

= 2'/9.3'

= 0.215

Angle of Attack A = 12.42 or 12° 25'

Load Factors

Load factors, the ratio of the total air load acting on an airplane to its gross weight, is expressed in factors of G, i.e. the normal gravitational effect on the planes gross weight. The factor is very significant in the structural design of the airplane. As the speed of the plane increases, the load factor increases. As a result, both steep turns (actually any maneuver) and wind gusts must be a concern of the pilot, so as to avoid exceeding the designed yeild points for the aircraft.

Stalling speed increases in proportion to the square root of the load factor. Any airplane, within its yield limits, can be stalled at any speed. Since the load factor squares as the stalling speed doubles, tremendous loads are created when a plane is stalled at high air speeds. Conventional airplanes can have the structural design strength exceeded in a snap maneuver (by intentionally stalling at entry speed). The maneuvering speed, the maximum speed at which a plane can be safely stalled is approximately 1.7 times its normal stalling speed.

Problem:

If your plane has a normal stalling speed of 45 mph (load factor 2G), Determine its maneuvering speed. What is the load factor at 90 mph?

Maneuvering speed = 1.7 times Normal Stalling Speed

= $1.7 \cdot 45$ mph

= 76.5 mph

Load Factor at 90 mph (Double normal 45 mph Stall Speed) = (Load Factor @ 45 mph)^2

= $(26)^2$

= 4 G's

Steering

Novice pilots frequently think of steering an airplane in a manner similar to driving a car, or perhaps a boat. He fails to realize that the execution of a turn is not accomplished by turning the rudder or using the ailerons. Rather the turn is accomplished by banking and allowing the horizontal and vertical components of the lift force to effect the turn. Banks of 60° or less provide adequate forces to accomplish any desired turn, however, due to the loss of vertical lift, the plane will lose altitude unless the amount of lift produced by the wing is increased by an angle or attack increase.

Coordinated use of the ailerons, rudder and elevator, produce the roll, offset the effects of adverse yaw and increase the angle of attack to increase the lift. A correctly made turn eliminates slipping or skidding and enhance the safety and economy of the flight (turn).

When coming into a turn from straight flight, the radius (R) of the circle (at the center of gravity of the plane) and the speed of travel (V) will produce a radial acceleration (A) equal to V^2/R,

Problem:

Calculate the radial acceleration (A) for an airplane flying at 95 mph as a 5000' turn is accomplished. (Note: 5280' = 1 mile).

Radial Acceleration $A = V^2/R$

$= (95\ \text{mph})^2/5000'$

$= 9025/.947\ \text{mi}$

$= 26.4\ \text{mph}$

At a given airspeed, the rate at which an airplane will turn depends upon the horizontal lift component. The angle of bank determines the rate of turn. The greater the rate of turn the greater the load factor increase. In this case, the load factor is the value which determines the total load (lbs) which the wing must support. A load factor of 1G for level flight becomes 2 G's in a 60° bank. The centrifigual force of the turn creates a "weight" equivalent to the gravitational weight. The increase in airspeed of the outer wing tends to cause an overbanking.

In compensating for the overbanking tendency, the degree of turn is usually accomplished accurately, however, the degree of bank is usually under estimated.

Problem:

As the load factor increases, the stalling speed increases. Steeply banked turns are impractical from both the load factor and the stalling speed standpoints. If the increased stalling speed is approximately 41% for a particular airplane in a 60° banked turn, caluclate the wing load weight and the banked turn stalling speed for a 7200 lb. plane whose normal stall speed is 40 mph as it accomplishes a 60° banked turn.

Wing Load at 60° Bank = 2 times Gravitational Weight
= 2 (G) • 7200 lbs.
= 14400 lbs.

Stall Speed at 60° Bank = Normal Stall Speed + 41% Increase
= 40 mph + 41% (40 mph)
= 141% (40 mph)
= 1.41 • 40 mph
= 56.4 mph

I. MATERIAL COST ESTIMATE

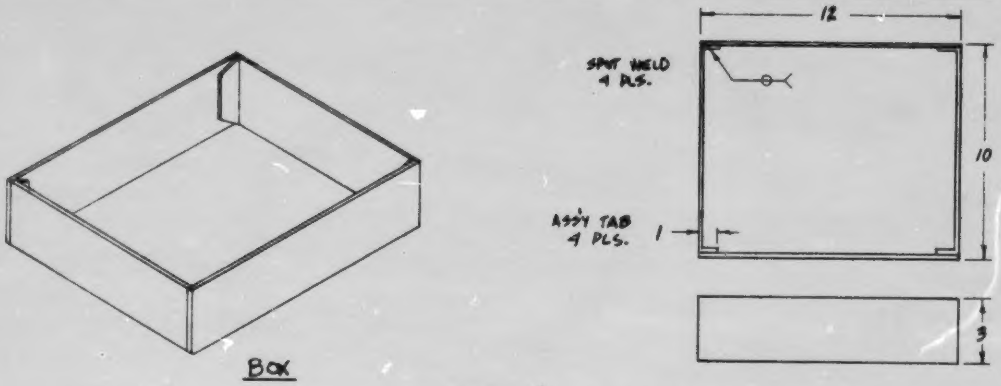
Problem: Determine the material cost to fabricate a given number of sheet metal parts.

Solution:

1. Determine the flat pattern of the part.
2. Calculate the net area of the part including waste.
3. Find the available sizes of sheet stock. Be sure it is of the correct thickness and temper.
4. Compare the flat patterns with the sheet stock sizes to decide which one is the most practical.
5. Divide the number of parts required by the number of parts per sheet to determine the number of sheets. Round up to a whole sheet.
6. Multiply the number of sheets by the price per sheet to equal the total material cost.

Example: Determine the material cost to fabricate 250 sheet metal boxes with the specifications shown in Figure #1. When working on the flat pattern, be sure to include assembly tabs to facilitate box construction. See Figure #2. Square off the shape of the flat pattern to a simple rectangle. Use this as the area of material per part. Now, look at the three available sizes of sheet stock. (Figure #3) Sheet one is too small, sheet three seems to accomodate exactly 3 widths; but room must be allowed for shearing or a saw kerf. Therefore, in this limited case, sheet two would be the correct choice. Each of these sheets have been assumed to be of the proper thickness and temper.

Since we can only make one part per sheet, we will need 250 sheets @ 1.22/sht. Total material cost for this job = $305.00.


12
SPOT WELD
4 PLS.
10
ASS'Y TAB
4 PLS.
1
3
BOX

NOTES:

1. ALL DIMENSIONS IN INCHES
2. CORNER RADII .04 MAX.
3. MATERIAL: .060" THK 6061
T6 ALUMINUM

FIGURE #1 SCALE: 1/4" = 1"

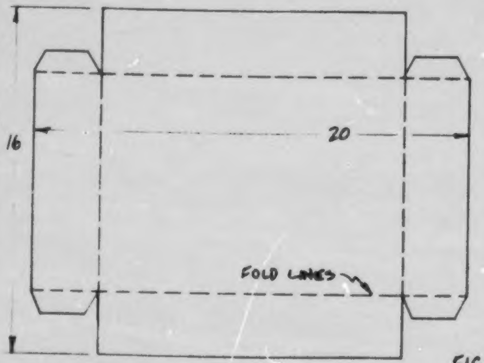

16
20
FOLD LINES

FIGURE #2 SCALE: 1/4" = 1"

AVAILABLE SHEET STOCK

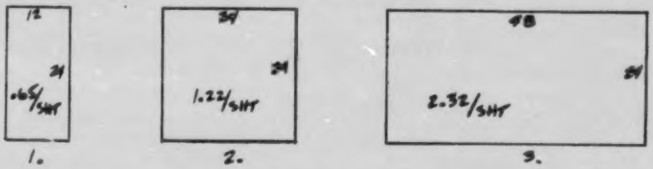

12
24
.65/SHT
1.
36
24
1.22/SHT
2.
48
24
2.32/SHT
3.

FIGURE #3 SCALE: 1/16" = 1"

SCALE AS SHOWN CK BY	MATERIAL COST ESTIMATE I	DALLAS COUNTY COMMUNITY COLLEGE DISTRICT	NAME R. PUCCI DATE 2/82 GRADE

II. QUADRATIC SAND BOX

Problem: You find **that** after a large job is over, a stockpile of eight foot long 2 x 8's are left over. You decide to utilize the material with as little waste as possible and build sand box frames. You also decide that they should cover an area no greater than 13.7 Sq. Ft. If two pieces of stock are used for each box, how long should the sides be?

Solution: Since the two sides of this rectangle have a total length of 8, they can be expressed as "x" feet long and (8 - x) feet long, since

$$x + (8 - x) = 8$$

Because the area of a rectangle equals the product of two adjacent sides, we can state: $x(8 - x) = 13.7$

or $8x - x^2 = 13.7$

By manipulating this expression we can put it in the basic form:

$$ax^2 + bx + c = 0$$

Thus: $x^2 - 8x + 13.7 = 0$

Since this expression is not easily factored it may be solved graphically. First, let another variable "y" equal zero:

$$y = 0$$

Therefore: $y = x^2 - 8x + 13.7$

By picking random but convenient values for "x", we can generate sets of ordered pairs.

Next, plot these points on the x, y axes. By drawing a smooth curve through these guiding points we should have approximated the location of all the ordered pair coordinates for the equation:

$$y = x^2 - 8x + 13.7$$

But we want the values that cause the expression to equal **zero**. So, by graphing y = 0 and finding the points that satisfy both equations, we will find our answer.

See Figure 1

There are two roots. They correspond to the two lengths when the eight foot piece is cut. They are Approx.

5.55, 2.45

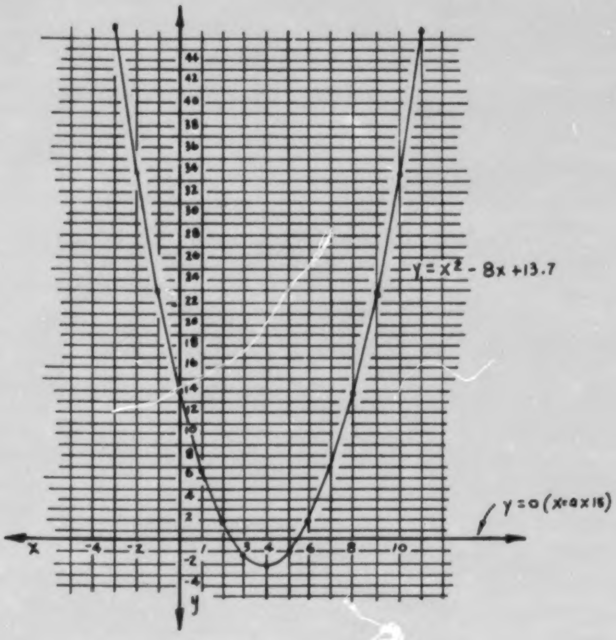


FIGURE # 1

TABLE

X	Y
-3	46.7
-2	33.7
-1	22.7
0	13.7
1	6.7
2	1.7
3	-1.3
4	-2.3
5	-1.3
6	1.7
7	6.7
8	13.7
9	22.7
10	33.7
11	46.7

SCALE NONE
CK BY

QUADRATIC SAND BOX II

DALLAS COUNTY COMMUNITY COLLEGE DISTRICT

NAME
DATE
GRADE

III. HARDWARE MATRIX

Problem: You've been assigned to organize the hardware stock in a sheet metal shop before a new project begins. Some existing stock may be applicable, some may not. Among the types of hardware are various combinations of finishes, sizes, and materials. A multidimensional chart or matrix can help here.

Solution: A four demensional matrix will suffice here. Limited by two dimensional paper for our chart, sub-charts defining "levels" in the four dimensions will be required. If five labels were necessary to describe each piece of hardware, then a five dimensional matrix would be used - 'n' labels → 'n' dimensional matrix. Of course pictures don't work after three!

For example, if six bits of information were required to totally describe a certain variety of bolt, then a set of 6 coordinates in letters or numbers would be written.

Ex. (9, 5, 2, 6, 3, 3)

Each coordinate represents an item in a list. The '9' may be the ninth available color, the '5' may be the fifth available thread size and so on.

The four 'dimensions' or charts will have 12 available choices each. Some 'slots' may not have assigned values and some combinations will not be valid since a minimum number of coordinates is desired and some descripters

	COOR.	SIZE	KNURL / HEAD TYPE	FINISH / MATERIAL	LENGTH
	0	—	—	—	—
SCREWS	1	4-40	SOCKET	STAINLESS	1/16"
	2	6-32	HEX	BRASS	1/8"
	3	8-32	PAN	COPPER PLATE	3/16"
	4	10-32	ROUND	TIN PLATE	1/4"
WASHERS	5	4-40	KNURLED	LEAD	5/16"
	6	6-32		PLASTIC	3/8"
	7	8-32			7/16"
	8	10-32			1/2"
RIVETS	9	1/8 DIA.			9/16"
	10	1/4 DIA.			5/8"
	11	3/16 DIA.			11/16"
	12	1/2 DIA.			3/4"
		(A)	(B)	(C)	(D)

- GENERAL FORM : (A,B,C,D)
- EXAMPLE : A 4-40, SOCKET HEAD, COPPER PLATED SCREW, 9/16" LONG WOULD BE (1,1,3,9)
- RULES FOR THIS CHART WOULD INCLUDE THAT THE FOURTH COORDINATE OF ANY WASHER = 0.
- BEWARE OF ERRONEOUS COMBINATIONS.

SCALE NONE CK BY	HARDWARE MATRIX III	DALLAS COUNTY COMMUNITY COLLEGE DISTRICT	NAME DATE	GRADE

have been combined.

See Drawing

IV. SUN'S ANGLE CALCULATION

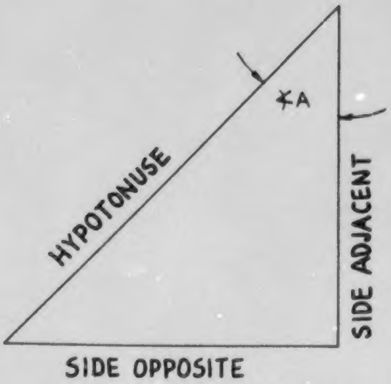
Problem: You live in a small mountainous town and wish to build a house. In order to determine design factors, such as the size of overhang required, the sun's angle of incidence must be known. Since no book can accurately predict this incidence angle for every point on the earth, a method of local measurement may be necessary.

Solution: This method must be used at different times of the day and also at different times of the year to be useful.

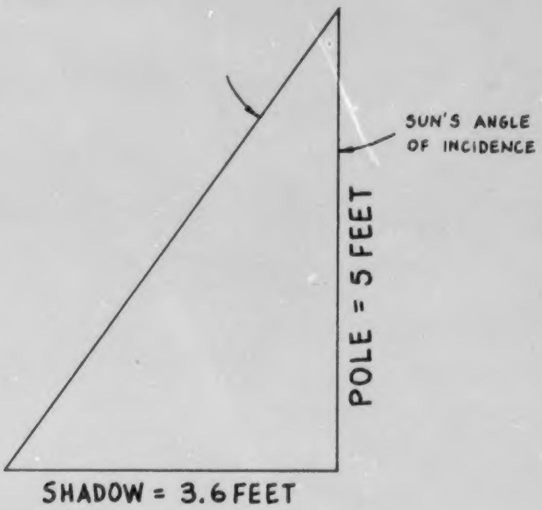
A calculation by trigonometry will solve this easily.

The materials needed will be a rod or pole, and a tape measure. First, drive the pole into the ground, leaving a convenient distance out of the ground such as 5 feet. The pole must be at 90 to the ground for us to use a convenient Trig. function. At any given time, the pole will cast a shadow. The length of the shadow will be the second leg in a right triangle. When comparing two legs (not the hypotenuse), the tangent function is used. Tangent of an angle equals the opposite side's length divided by the adjacent. (Figure 1)

Applying this to the pole and shadow we may solve for this given example. (Figure 2)


HYPOTONUSE
∡A
SIDE ADJACENT
SIDE OPPOSITE

$\tan \measuredangle A = \dfrac{\text{SIDE OPPOSITE}}{\text{SIDE ADJACENT}}$


SUN'S ANGLE OF INCIDENCE
POLE = 5 FEET
SHADOW = 3.6 FEET

$\text{TANGENT OF SUN'S } \measuredangle = \dfrac{\text{LENGTH OF SHADOW}}{\text{LENGTH OF POLE}} = \dfrac{3.6}{5}$

$\text{TAN } \measuredangle = .7$

$\text{SUN'S } \measuredangle \text{ OF INCIDENCE} = 35°$

SCALE NONE
CK BY
SUN'S ANGLE CALCULATION IV
DALLAS COUNTY COMMUNITY COLLEGE DISTRICT
NAME
DATE
GRADE

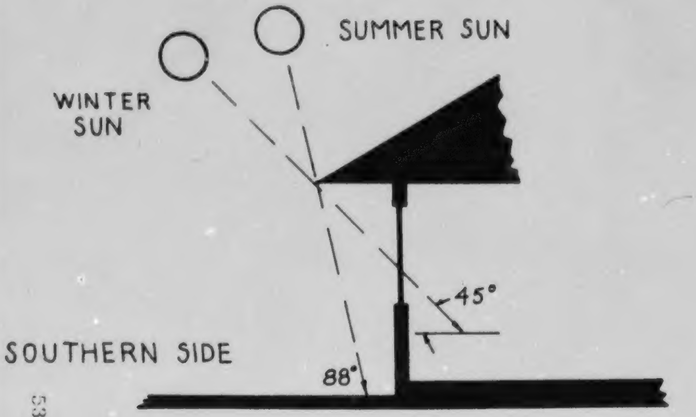

SUMMER SUN
WINTER SUN
45°
SOUTHERN SIDE
88°
534

FIGURE #1

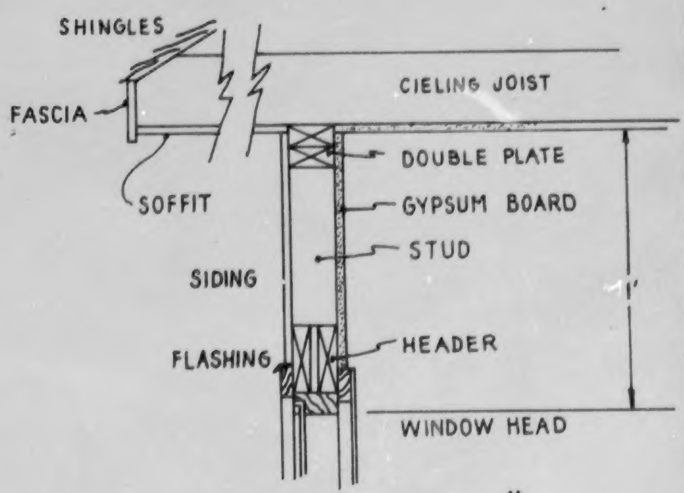
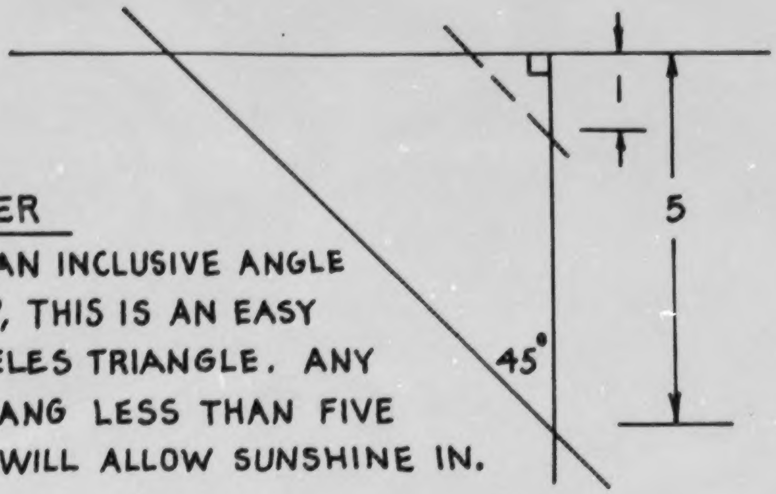

SHINGLES
CIELING JOIST
FASCIA
DOUBLE PLATE
SOFFIT
GYPSUM BOARD
STUD
SIDING
1'
HEADER
FLASHING
WINDOW HEAD

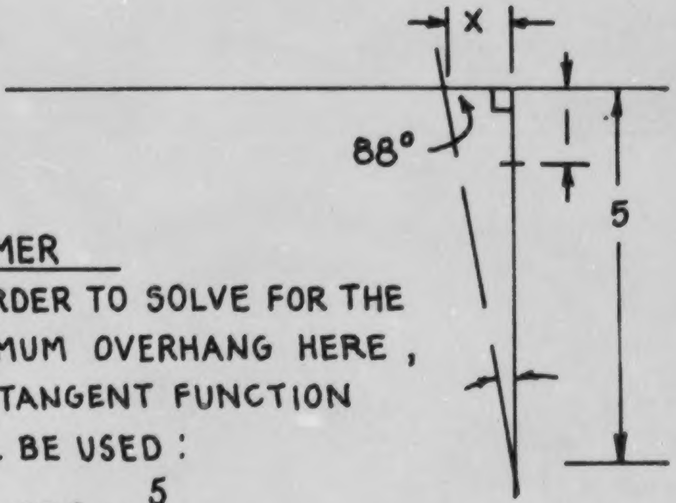
FIGURE #2

SCALE NONE CK BY	SUN'S ANGLE IV.	DALLAS COUNTY COMMUNITY COLLEGE DISTRICT	NAME PUCCI DATE GRADE



WINTER

WITH AN INCLUSIVE ANGLE OF 45°, THIS IS AN EASY ISOSCELES TRIANGLE. ANY OVERHANG LESS THAN FIVE FEET WILL ALLOW SUNSHINE IN.



SUMMER

IN ORDER TO SOLVE FOR THE MINIMUM OVERHANG HERE, THE TANGENT FUNCTION WILL BE USED:

$$\mathrm{TAN}\ 88^\circ = \frac{5}{X}$$

$$X = \frac{5}{\mathrm{TAN}\ 88^\circ}$$

$$X = .174$$

FIGURE # 3

SCALE NONE
CK BY
SUN'S ANGLE IV.
DALLAS COUNTY COMMUNITY COLLEGE DISTRICT
NAME Pucci
DATE
GRADE

Many readings should be made and documented to assure proper measurement.

V. PROJECT MATERIAL PLANNING

Problem: Determine the amount of material needed to manufacture several castings in various quantities.

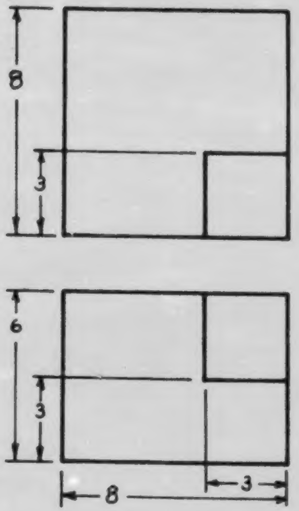
Solution:

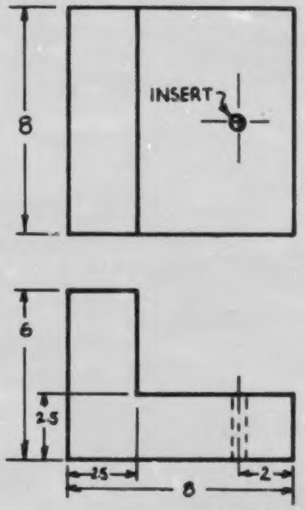
1. Develop a bill of materials for each part. Be sure to include some type of part identification such as part number or letter. Also, identify any hardware or finish material.
2. Multiply the quantity of each item by the number of parts required in order to determine total requirements of each item. Do this for each part.
3. Add quantities of like items to determine total project requirements. Add overage where required.

Example: Build the following quantities of castings:

Part #	Quantity
1	100
2	50
3	150
4	250
5	75

See drawings for multiple views and bills of material





PART # 1

VOLUME = 8 X 8 X 6
- 3 X 3 X 3
V = 384 − 27 = 357 sq. units
QUANTITY = 100
MATERIAL = LEAD

TOTAL MAT. = 35,700 sq. units LEAD

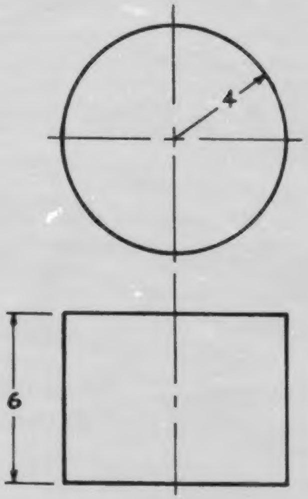
PART#2

VOLUME = 6 X 2.5 X 8
+ 5.5 X 2.5 X 8
V = 120 + 110 = 230 sq. units
QUANTITY = 50
MATERIAL = IORN
HARDWARE = INSERT (1 PER)
TOTAL MAT. = 11,500 sq. units IRON, 50 INSERTS

PROJECT REQUIRMENTS

80. 910 sq. units LEAD, 28,127.5 sq. units IRON, 80,000 sq. units COPPER, 50 inserts.

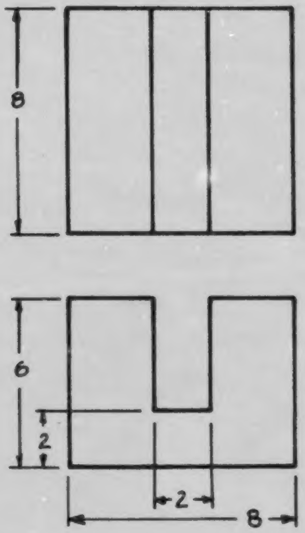
SCALE NONE	PROJECT MATERIAL PLANNING V	DALLAS COUNTY COMMUNITY COLLEGE DISTRICT	NAME	
CK. BY			DATE	GRADE



PART # 3

VOLUME = $\pi\ (4)^2 \times 6$
= 50.24 X 6
V = 301.4 sq. units
QUANTITY = 150
MATERIAL = LEAD

TOTAL MAT. = 45,210 sq. units LEAD

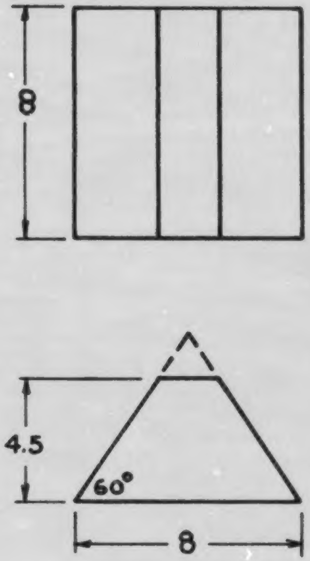


PART # 4

VOLUME = 8 X 8 X 6
− 2 X 4 X 8
V = 384 − 64 = 320 sq. units
QUANTITY = 250
MATERIAL = COPPER

TOTAL MAT. = 80,000 sq. units COPPER

SCALE NONE
CK BY
PROJECT MATERIAL PLANNING V
DALLAS COUNTY COMMUNITY COLLEGE DISTRICT
NAME
DATE
GRADE



PART# 5

VOLUME = $\frac{1}{2} \times 8 \times \sqrt{48} \times 8$

= 221.7 sq. units

QUANTITY = 75

MATERIAL = IRON

TOTAL MAT. = 16627.5 sq. units IRON

SCALE NONE	PROJECT MATERIAL PLANNING ✓	DALLAS COUNTY COMMUNITY COLLEGE DISTRICT	NAME	
CK BY			DATE	GRADE

VI. PACKAGE SYSTEM

Problem: Design a package system of cylindrical metal containers and rectangular cardboard boxes to fit shelves that can accomodate a height of six inches. There shall be 24 containers per box and each container shall have a net volume of one quart.

Solution: Make a sketch of the possible arrangments twenty-four cylinders can make. (By reducing the total quantity to its 'prime factors' we may easily generate the four possible variations.)

$$24 = 2 \times 2 \times 2 \times 3$$

1 x 24 2 x 12 3 x 8 4 x 6

See Figure #1

Since we must make a cardboard box for the arrangment chosen, we must be practical about the shape. Because a square happens to have the highest volume to perimeter ratio, the arrangment having factors closest to equal will require a box using the least material. Therefore, the box will be four container's diameter by six diameters.

Next, the diameter of one container must be calculated. We know it must have a volume of one quart and a height of six inches. These two bits of information must be in common terms to be useful. The only volume to square unit conversion I know is metric; $1\ cm^3 = 1\ ml$. So, I will use the following steps to arrive at a cubic inch equivalent for one quart.

$$1\ quart = 946\ ml$$

$$1\ ml = 1\ cm^3$$

$$\therefore 1\ quart = 946\ cm^3$$

$$1\ inch = 2.54\ cm$$

$$1\ inch^3 = 16.39\ cm^3$$

$$\therefore 1\ quart = 57.7\ in^3$$

Knowing the volume of a cylinder equals π times the radius of the cylinder times the height, and knowing the values for two of the three variables in this formula, we can derive the diameter of the cylinder required. See Figure #2.

$$\text{Volume of Cylinder} = \pi\ r^2\ h$$

Solving for the variable r =

$$r = \frac{v}{\pi h}$$

Substituting known values =

$$r = \frac{57.7}{6\pi}$$

Solution: $r = 1.75''$

$2r = D$

$D = 3.5''$

The box dimensions from Figure #1 (in diameters) may be expanded to inches:

$40 = 14''$ $6D = 21''$

Summary: 24 cylinders with a 3 - 5" diameter and a height of six inches will be arranged in four rows of six in a box with dimensions 6" x 14" x 21". (Material thickness must be added for exterior dimensions.) See Figure #3.

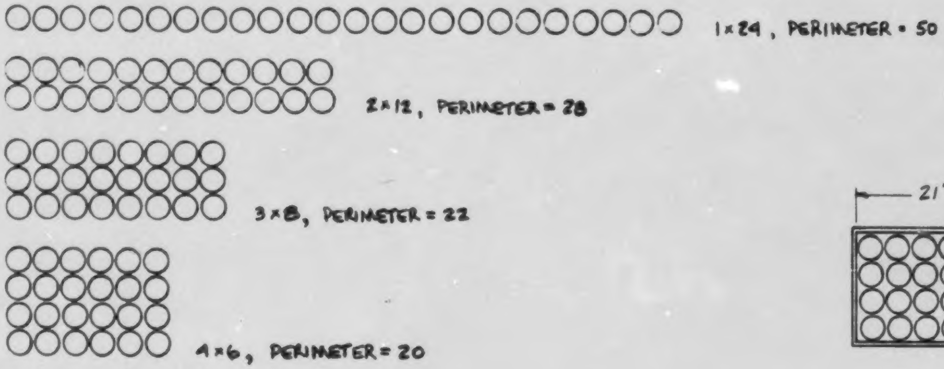
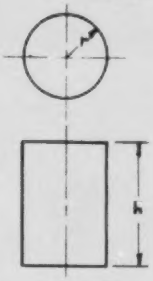

1×24, PERIMETER = 50
2×12, PERIMETER = 28
3×8, PERIMETER = 22
4×6, PERIMETER = 20

FIGURE #1



VOLUME OF A CYLINDER = $\pi r^2 h$

FIGURE #2

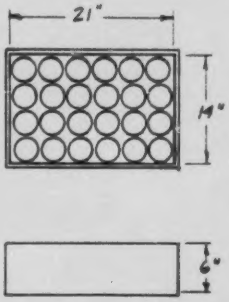


FIGURE #3

SCALE NONE CK BY	PACKAGE SYSTEM VI	DALLAS COUNTY COMMUNITY COLLEGE DISTRICT	NAME R. PUCCI DATE 2/82 GRADE

VII. WEIGHT CONVERSION

Problem: You are the manufacturing representative on a project to build a retro rocket housing on a space vehicle. Because the engine in that housing burns at extremely high temperatures, it must be made of a high density, exotic, and very expensive alloy. Its final shape and size are not yet finalized. The end product shape will be decided upon after multiple wind tunnel tests. The designer's first need is to know the weight of the final design for balance calculations.

Solution: Since the shape must be constantly modified for sind tunnel tests, the original model should be made of an easily formed, relatively inexpensive material such as plastic or wood. If the material used has a known density and a measured weight, we can solve for its volume. Then knowing the desired volume, and the density of the actual material to be used, we can calculate the final weight of the housing.

Using a plastic material:

Given:

Density = 8 lbs. per cubic foot

Design #1 weight = 4.329 lbs.

Solve for Volume.

Since Density = weight/unit volume (lbs/ft)

V = weight/density

$$V = \frac{4.329\text{ lbs.}}{8\text{ lbs/ft}^3}$$

See Figure 1 for

$V = .541\text{ ft}^3$ Graphic Solution.

Knowing the density of the final material, and now the volume of the housing, we can solve for its final weight.

Again, since Density = weight/unit volume

Solving for weight: W = (Density)(Volume)

Given: Density of exotic material = 260 lbs/ft^3

Calculated volume = 541 ft.^3

Then: W = (260)(.541) See Figure #2

W = 140.66 lbs. For Graphic Solution.

Having gone through this once, we can recognize a relationship:

4.329 lbs plastic corresponds to 140.66 lbs exotic Mat. Since density is per unit volume another relationship can be seen:

8 lbs/ft^3 and 260 lbs/ft^3 are in the same ration as the weights above. Using the plastic weights as 'y' coordinates and the exotic material weights as 'x' coordinates, we can draw a line within which all points have as coordinates the various corresponding weights of plastic vs. exotic.

See Figure 3.

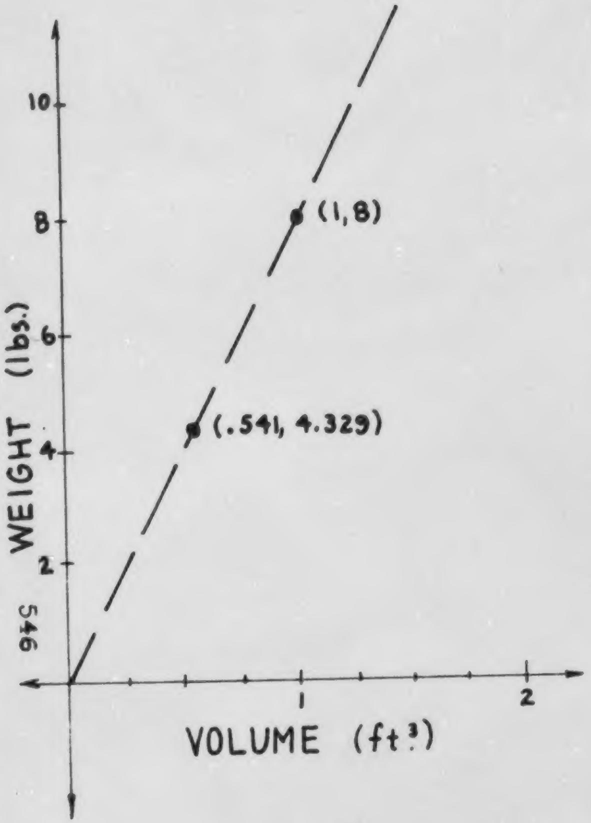
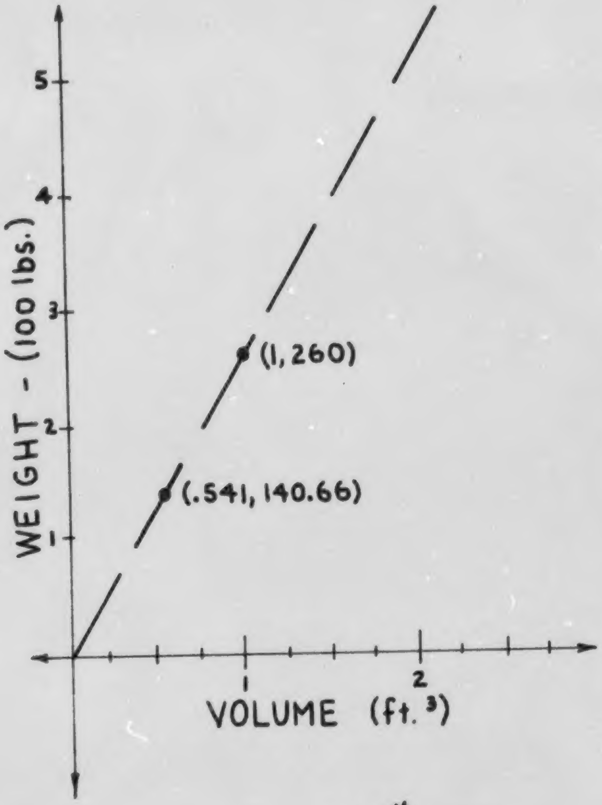


FIG.#1



FIG#2

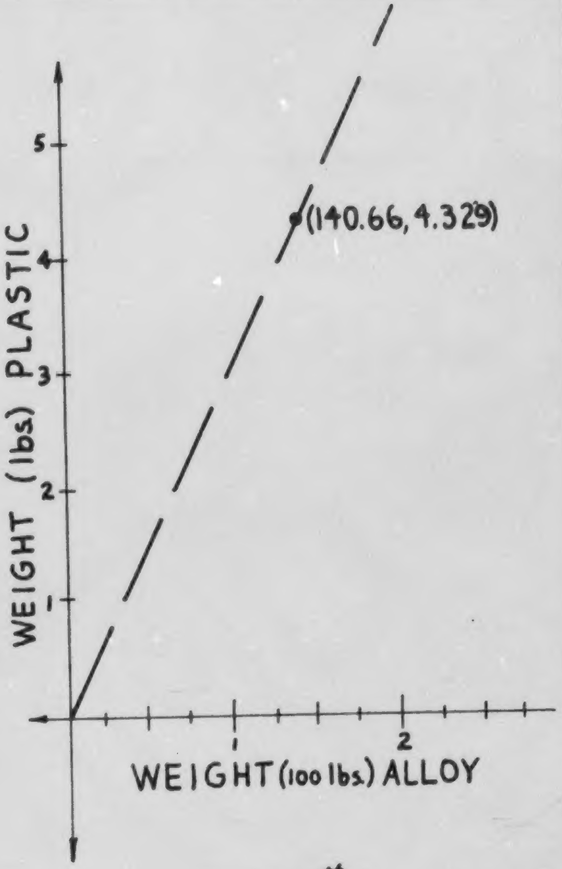


FIG. #3

SCALE NONE CK BY	WEIGHT CONVERSION CHARTS	DALLAS COUNTY COMMUNITY COLLEGE DISTRICT	NAME	
			DATE	GRADE

VII

VIII. OVERHANG CALCULATION

Problem: Determine the size overhang required on the southern exposure of a house in order to shade the windows in summer yet allow sunchine to enter in winter.

Solution: There are multiple factors affecting this problem. The sun's angle of incidence varies with both the season and position on the earth, namely latitude. The height of the walls and windows involved also add factors to the problem.

In this example, we will not deal with design or aesthetics. We will only deal with the geometry involved. Therefore, the solution will merely be a point in space to which any designer or architect may connect his or her favorite overhang.

For our example, let the winter or minimum angle of the sun equal 45 and let the summer or maximum angle equal 88 from the horizon. (Figure 1)

Let the window be four feet in height and located one foot below an eight foot interior ceiling. The overhang, in this case, will be constructed from the same ceiling joist. (Figure 2)

By reducing the two extreme cases to simple right triangles, we can solve for the overhang range that lets sunshine penetrate the window in the winter and also solve for the minimum overhang required in summer to keep the sun out. (Figure 3)

Any overhang within both parameters will satisfy the requirements of this specific situation. Any change of window dimension, geographic location, or house orientation would require a new calculation.

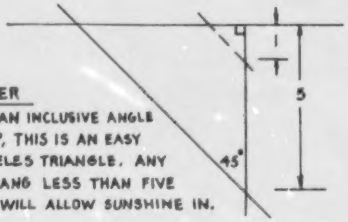
Now, we must decide at what time of year the sun should be allowed to shine into the window. If, for example, September is the month of changing temperature for this region, we can look on a chart for the angle of incidence of the sun during that month. If it is 64 , then solving a similar problem with that angle would produce the desired overhang.

From Figure 3 we can see that a simple formula may now be used:

$$\text{Length of overhang} = \frac{\text{Distance from ceiling to window sill}}{\text{Tangent (sun incidence) when sunlight is desired to start.}}$$

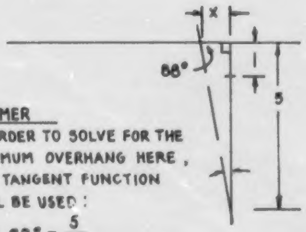
$$\text{So:}\quad \text{overhang} = \frac{5}{\text{Tan } 64}$$

$$= 2.439'$$



WINTER

WITH AN INCLUSIVE ANGLE OF 45°, THIS IS AN EASY ISOSCELES TRIANGLE. ANY OVERHANG LESS THAN FIVE FEET WILL ALLOW SUNSHINE IN.



SUMMER

IN ORDER TO SOLVE FOR THE MINIMUM OVERHANG HERE, THE TANGENT FUNCTION WILL BE USED:

$\tan 88° = \frac{5}{X}$

$X = \frac{5}{\tan 88°}$

$X = .174$

FIGURE # 3

SCALE NONE CK BY	OVERHANG PROJECTION VIII	DALLAS COUNTY COMMUNITY COLLEGE DISTRICT	NAME DATE GRADE

IX. SIMULTANEOUS EQUATION

Problem: You work at a mill that produces many alloys. A customer wants to buy four tons of bronze for a run of candlesticks he plans to release. He can pay a maximum of 52 dollars per pound. You stock copper at 78 dollars per pound and tin at 8 dollars per pound. What percentage of copper can you offer him for his price? Also the amounts of copper and tin needed for the order must be calculated.

Solution: Let 'x' equal the number of pounds of copper required for the job.

Let 'y' equal the number of pounds of tin.

If the copper is $78 and the tin is $8 per pound then the following expression can describe the 8000 pound order at $52 per pound:

$$78\,x + 8\,y = 52\,(8000)$$

Using pounds only, we may write the following:

$$x + y = 8000$$

Having two expressions containing the same two unknowns, we may solve by simultaneous equations:

$$\begin{aligned} \text{Since } x + y &= 8000 \\ \text{Then: } 8x + 8y &= 64000 \end{aligned}$$

$$\begin{array}{rrcr} \text{Now:} & 78x + 8y & = & 416000 \\ \text{Subtract} & \underline{8x + 8y} & \underline{=} & \underline{64000} \\ & 70x & = & 352000 \\ & x & = & 5028.57 \text{ lbs} \qquad \text{Copper Required} \end{array}$$

Substituting for x:

$$\begin{aligned} 5028.57 + y &= 8000 \\ y &= 2971.43 \text{ lbs} \qquad \text{Tin Required} \end{aligned}$$

Now, to answer the question of copper percentage:

$$\frac{5028.57}{8000} = 62.86\%$$

GENERAL INFORMATION FOR SOLVING PHARMACOLOGY PROBLEMS

Pharmacology is the study of medicines and their actions upon the body. Today there are three major systems of weights and measures in which pharmaceuticals (drugs) are prescribed, compounded and administered. Students in the allied medical professions must be able to work problems within each system and to convert from one system to another.

The Apothecaries system, of Greek origin, is the oldest of the three systems. The smallest unit of dry measure is the grain, other units are the scruple, which is very seldom used, the dram, the ounce, and the pound. The smallest liquid measure is the minim, which is the amount of water that weighs one grain. Other liquid units are the fluidram, the fluidounce, the pint, the quart, and the gallon. The following symbols or abbreviations may be used for these units.

grain - gr.
scruple - sc. or ℈
dram - dr. or ʒ
ounce - oz. or ℥
pound - lb. or #

minim - m
fluidram - dr. or fʒ
fluidounce - oz. or f℥
pint - pt. or 0
gallon - C.

Roman numerals are used generally to express quantity with this system, except for fractions and quantities over 50. The symbol, $\overline{ss}$, or ss, is frequently used to express the fraction ½. Examples of Apothecary measures are:

gr., iii, ʒss, and Ov.

To facilitate solving of problems, the student may substitute Arabic numerals for the Roman numerals. The most frequently used Roman numerals are:

I - 1 X - 10
V - 5 L - 50

The metric system is fast replacing the Apothecaries system in general usage. This system was established to be a universal system for those engaged in scientific, industrial, and educational careers. In America today, there is a move to make metric measures household words. The most common units of dry measure used in medicine are the milligram, the gram, and the kilogram. The most commonly used liquid measures are the cubic centimenter, the mililiter, and the liter. The abbreviations are listed below.

miligram - mgm. or mg. cubic centimeter - c.c.
gram - Gm. or gm. mililiter - ml.
kiligram - Kg. or kg. liter - L. or 1.

Arabic numerals precede the unit in this system, for example, 2 gms., or 6 ml.

Houschold measures are also used in medicine, for liquids such as cough syrups which are to be taken at home. The measures are used in the hospitals as well. Household utensils vary somewhat in size making this system the least accurate of the three.

The most commonly used household measures are the teaspoon, the tablespoon, and the cup.

Some drugs are compounded only in units or miliequivalents (mEq). These drugs are usually not ordered in or converted to any other system of measurement.

Pharmacists use exact equivalents when compounding prescriptions. Nurses use approximate equivalents when administering drugs or pharmaceuticals which are already compounded. Below is a list of equivalents which may be used to solve pharmacology problems.

APPROXIMATE EQUIVALENTS

Dry Measure	Liquid Measure
½ gr = 5 mg	1 ml = 1 cc
½ gr = 30 mg or 32 mg	1000 ml = 1 L
1 gr = 60 mg or 64 mg or 65 mg	15 m = 1 ml
1½ gr = 100 mg or 90 mg	60 m = 1 dram
15 gr = 1 gm	8 drs = 1 oz
2.2 lbs = 1 kg	32 ml or 30 ml = 1 oz
	1 tsp = 1 dr or 4 ml or 5 ml
	3 tsp = 1 Tbsp
	2 Tbsp = 1 oz or 30 ml or 32 ml
	15 drops = 1 ml
	8 ounces = 1 cup

Ratio and proportions may be used to convert from one system to another. An exercise is provided for your practice.

Give equivalent in blank provided below:

1) 2 teaspoons = __________ounces __________ml.
2) 1 Gram = __________Kilogram __________mg.
3) 4 ounces = __________tablespoon __________dram
4) 10 c.c. = __________teaspoon __________minims
5) 2 tablespoons = __________ounces __________c.c.
6) 4 c.c. = __________drams __________gtt.
7) 15 grains = __________Gm. __________mg.
8) 1 c.c. = __________minim __________gtt.
9) 2 tablespoons = __________ounces __________ml.
10) 2000 c.c. = __________quarts __________cups
11) 1 ounce = __________c.c. __________teaspoons
12) 1 dram = __________m __________teaspoons
13) 2 ounces = __________tbsp. __________teaspoons
14) 1 pint = __________Liter __________cups
15) 30 m = __________ml __________gtts.
16) 4 drams = __________c.c. __________ounces
17) 1 ounce = __________minims __________tsp.
18) 1 cup = __________pint __________c.c.
19) 2 ft. oz. = __________c.c. __________tbsp.
20) 30 grains = __________Grams __________miligrams

ANSWERS TO PRACTICE EXERCISE

1)	.25 or .3 ounces	8 or 10 ml.
2)	0.001 Kilogram	1000 mg.
3)	8 tablespoon	32 dram
4)	2 teaspoon	150 or 160 minims (m)
5)	1 ounce	30 or 32 c.c.
6)	1 dram	60 or 64 gtt.
7)	1 Gm.	1000 mg.
8)	15 or 16 minim	15 or 16 gtt.
9)	1 ounce	30 or 32 ml.
10)	2 quarts	8 cups
11)	30 or 32 c.c.	6 or 8 teaspoons
12)	60 m	1 teaspoon
13)	4 tbsp.	12 teaspoons
14)	0.5 Liter	2 cups
15)	2 ml.	30 gtts.
16)	16 or 20 c.c.	0.5 ounces
17)	450 or 480 minims	6 or 8 tsp.
18)	0.5 pint	240 or 250 c.c.
19)	60 c.c.	12 tbsp.
20)	2 Grams	2000 miligrams

Ratio and proportion is the most common mathematical method used to solve pharamacology problems. The ratios are in the amount of medication (dosage) contained in a vehicle (tablet, capsule, or measured amount) of solution. The proportions are between the amount/vehicle which has been ordered and the amount/vehicle which is in stock, that is, as the pharmaceutical house packages the drug. For example, Vitamin C is marketed in 100 mg., 250 mg., and 500 mg. tablets.

When solving problems which use only dry measures, the following formula is used.

D	:	H	::	X	:	1
desired dose		dose on hand "stock"		? number of tablets or capsules needed		1 tablet or capsule

An alternate expression of this formula is:

$$\frac{D}{H} = \frac{X}{1}$$

Sample problems using the above formula are worked for you. Aspirin is an example of a drug which may be ordered in metric or apothecaries units. A problem using metric measures, a problem using apothecaries measures and a problem requiring conversion from one system to another will be worked.

EXAMPLE 1: Aspirin 650 mg. has been ordered. The medication on hand is labeled Aspirin 325 mg. How many tablets will be needed to give the required dose?

Step 1: state formula. D:H::X:1 $\frac{D}{H} = \frac{X}{1}$

Step 2: substitute appropriate numbers for letters in formula.

650 mg.: 325mg.::X:1 $\frac{650\text{mg}}{325\text{mg}} = \frac{X}{1}$

Step 3: solve for X Corss mult. for X

labels may be dropped

$325X = 650$

$\frac{325X}{325} = \frac{650}{325}$

$325X = 650$

$X = 2$

X = 2 tablets (go back to equation for label if needed.)

EXAMPLE 2: Aspirin gr.x. have been ordered for your patient. The medication in stock is labeled gr. v. How many tablets will be needed to give the correct dose?

Step 1: state formula D:V::X:1

Step 2: substitute appropriate numbers for letters in formula. Arabic numerals may be substituted directly for ease in calculations. gr.10:gr.5::X:1 tablet

Step 3: solve for X

10 :5 :: X :1

5X = 10

X = 2 tablets

EXAMPLE 3: Aspirin gr.x has been ordered. The bottle in the drug cabinet is labeled 325 mg. How many tablets would be needed to give the correct dose?

Note: before this problem can be solved it will be necessary to convert either grains x to X miligrams or 325 mg. to grains X. Use a ration and proportion to set up the conversion using one of the following equivalents: grains 15 = 1 gram, or 60 mg. = 1 grain.

gr. $\overset{(10)}{x}$: X mg.:: gr 1: 60 mg.
X = 60 · 10 = 600 mg.

The 600 mg. is now substituted for gr. x in the problem.

Step 1: state formula D:H::X:1
Step 2: substitute numbers for letters in formula
600mg.: 325mg.:: X tablets : 1 tablet
Step 3: solve for X
325X = 600
X = 1.846 tablets or 2 tablets rounded to ½ tablet.

The above problem illustrates the reason a nurse may use an approximate equivalent when giving prepared medication, either 60, 64, or 65 mg., could have been used as equal to grains 1. Had 65mg. been used in the example, the problem would have worked out evenly to 2 tablets. In these exercises round off all answers to the nearest ½ tablet or pill, and to the nearest one (1) capsule, because capsules are never broken.

EXERCISES

1. Atropine sulfate tablets gr. 1/300 were ordered for your patient. The stock bottle read gr. 1/150. How many tablets are needed for the correct dose?

Ans 1/2 tablet

2. You are to give a drug for which the dose is gr. 1/150. The label reads 0.2 mg. How many tablets will be required for the proper dose?

Ans. 2 tablets

3. The label reads Thyroid 30 mg. How many tablets are required to give gr. ¼?

Ans 1/2 tablet

4. Diethylstilbesterol was labeled 0.25 mg. The order was for 0.5 mg. How many tablets are needed to give the proper dose?

Ans 2 tablets

5. You are to give 1 gm. of a drug. The available tablets are labeled 250 mg. How many tablets are needed to give the appropriate dose?

Ans 4 tablets

6. The physician has ordered that a drug be given in three doses so that the patient receives a total of 0.75 gm. per day. You have on hand 125. mg. tablets. How many tablets will be required for each dose?

Ans 2tablets

7. The physician has ordered 4.5 mg. of a drug daily. The pharmacy sends a bottle of 1.5 mg. tablets. How many tablets will be needed for the correct dose?

Ans 3 tablets

8. The physician has ordered Keflex 0.5 gm. On hand are 125 mg. capsules. How many capsules will be needed for the correct dose?

Ans 4tablets

9. You are to give 0.6 gm. of a drug. The label reads gr.v. How many tablets will be needed to give the correct dose?

Ans 2tablets

10. The label of a medication reads 500 mg. How many tablets will be needed to administer gr. xv.?

Ans 2tablets

Ratio and proportion may also be used to calculate dosages of liquid measure whether the medication will be administered orally or parenterally (by injection). Liquid medication actually contains a solute in a solvent. The following formula is used.

D :	H =	X :	V
Desired dose	Dose on hand	Amount of solution required	Amount of solution on hand

EXAMPLE 1: Demerol 100mg. has been ordered. The medication on hand is labeled 50mg./cc. How many c.c.'s will be needed to give the required dose?

Step 1: state formula

D:H = X : V

Step 2: sutstitute appropriate numbers for letters in formula.

100 mg.: 50mg.: = X : 1 c.c.

Step 3: solve for x

100: 50 = X:1

50X = loo

$\frac{50X}{50} = \frac{100}{50}$

X = 2 ml. (be sure to label answers.)

EXAMPLE 2: Morphine sulfate gr. 1/6 is ordered. In the medicine cupboard are single dose vials labeled gr. 1/4 / ml. How many ml will be required?

Step 1: state formula D:H::X:V

Step 2: substitute appropriate numbers for letters in formula.

gr. 1/6 : gr. 1/4 :: X : 1

Step 3: solve for X

1/6 : 1/4 :: X : 1

1/4X = 1/6

$$\frac{1/4X}{1/4} = \frac{1/6}{1/4} = 1/6 \cdot 4/1 = 4/6 = 2/3$$

X = 2/3 ml.

EXAMPLE 3: A child is ordered penicillin suspension 250 mg. The suspension is needed to give the required dose (A c.c. is equal to 1 ml. and the two measures are used interchangeably although ml. is being used more often).

Step 1: state formula D:H::X:V

Step 2: substitute appropriate numbers for letters in formula.

250mg : 125mg :: X : 5 c.c.

Step 3: solve for X

250 : 125 = X : 5

125X = 1250

X = 10 c.c.

EXAMPLE 4: You are to give sodium luminal gr. vllss. The vial is labeled 1 gm./2ml. How many mililiters will be needed to give the correct dose?

Step 1: state formula

D:H::X:V

Step 2: substitute appropriate numbers for letters in formula.

gr 7.5: 1 gm.:: X ml.: 2 ml.

Step 2b: convert gr.7.5 to X gm.

7.5:X ‡ 15gr.: 1 gm.

X ÷ 0.5gm

or 1 gm. to gr.X

or direct equivalent
memorized 1 gm. = gm.15

Step 3: solve for X

0.5 gm: 1 gm:: Xml. 2 ml.

X = 1.0 or 1 ml.

or 7.5gv:gv 15 = Xml:2ml

15X = 15

X = 1 ml.

EXERCISES

1. The label reads atropine sulfate gr. 1/150 per ml. How much solution is needed to administer grains 1/300?

ANS: 0.5 ml

2. The label reads digitoxin 0.[illegible] mg. per 3 c.c. How much will be needed to administer 0.3 mg.?

ANS: 4.5 cc

3. Pediatric achromycin suspension contains 125 mg/5 c.c. How much is needed for a 150 mg. dose?

ANS: 6cc

4. Your patient is to receive 0.5 million units of Buffered Penicillin G every six hours for 12 days. How many ccs containing 200,000 units must be given at each dose?

ANS: 2.5 cc

5. Pediatric achromycin suspension is labeled 125mg. / 5 ml. You are to give a 25 mg. dose. How many drops would you give if a dropper calibrated to 16gtts/cc were used to measure the dose?

ANS: 16 gtts

6. You need to give 500 mg. orally of a drug which comes in a powder to be mixed. If the label reads 10 gms., add 118 ml of water to yield 200 ml. of suspension, how many c.c.'s will be required to give the correct dose?

ANS: 10cc

7. Penbretin 100mg. is ordered. You have an oral suspension containing 250mg./5ml. How many ml.'s will be required?

ANS: 0.5 ml

8. The physician has ordered potassium chloride (an electrolyte) 20 mEq p.o. daily. The pharmacy sends a bottle of potassium chloride liquid which contains 40 mEq per 10 c.c. How many c.c.'s will you give?

ANS: 5 cc

9. A liquid medication, Mellaril 0.2 gm., has been ordered for your patient. Your stock bottle is labeled 100 mg./ 5ml. How many ml. will be needed to give the correct dose?

ANS: 10 ml

10. Give a medication 0.5 gm. from a solution of chloral hydrate labeled 1 gm. = 20 ml.

ANS: 10 ml

11. You are to give 2 gr. of a drug. The solution you have on hand contains 30 mg. per 0.6 c.c. How many c.c.'s will you give? (60 mg = 1 gr.)

ANS: 2.4 cc

12. Your patient is to receive a liquid medication 0.25 mg. daily. You will be using liquid which is labeled 0.125 mg. 1 ml. How many ml. of the solution will your patinet receive?

ANS: 2 ml

13. The physician has ordered 0.06 mg. of a medication to be administered intramuscularly to a 3 year old. The ampule is labeled 0.2 mg per ml. How many minums will be given?

ANS: 4.5 m

Clark's rule is used to calculate the pedictric dosages of certain classes of medication. The students will most often see Clark's rules stated as follows:

$$CD = \frac{\underset{\text{[in pounds]}}{\text{Child's Weight (CW)}} \cdot \text{the average adult dose (AAD)}}{150 \text{ (the average adult weight)}}$$

The rule can also be expressed as a ratio and proportion.

CD : AD:: CW : AW
(X is usually substitued for CD in problem solving.)

EXAMPLE: The average adult dosage of a medication is 0.75 gm. How much would the child receive if the child weighed 50 pounds?

Step 1: state formula CD: AD:: CW : AW

Step 2: substitute appropriate numbers for letters in formula.

X: .75 gm.::50# : 150#

150X = 37.50

X = .25 gm.

1. Tommy, who weighs 60 lbs., is to receive a drug, which the adult dose is 500 mg. How much will Tommy receive?

ANS: 200 mg

2. Rodney,who weighs 45 lbs., is to receive a drug. The adult dose is 250 mg. How many mg. will Rodney receive?

ANS: 75 mg

3. Sharon, a ten year old, weighs 15 Kg. She is to receice a medication of which the adult dose is 50 mg. How many mg. will Sharon receive?

ANS: 11 mg

Nurses are expected to be able to determine the flow rate of an intravenous (IV) solution which may be infusing over a period of hours. The calculation of flow rate is a three step process.

Step 1: $\frac{\text{amount of solution to be infused}}{\text{time in hours it is to infuse}}$ = ml. / hr.

Step 2: $\frac{\text{ml / hour}}{\text{60 min.}}$ = ml / min.

Step 3: Note: most infusion sets are calibrated to deliver 15 gtts. (drops) per mililiter. Pediatric sets usually deliver 60 gtts. per minute, and blood administratorion sets deliver 10 gtts. per minute. (If there is any question concerning the number of drops per minute, the package label will give the information.) The "drops per minute" is usually referred to as drop factor (df).

$\overset{\text{(ml)}}{\text{c.c.}}$ per mimute · 15 df = gtts./ min. (the flow rate)

EXAMPLE: Administer 1 liter (1000 ml.) of D-5-W over a 5 hour period, use an administration set with a df of 15.

Step 1: $\frac{1000}{5}$ = 200 cc/hr (Carry to three places)

Step 2: $\frac{200}{60}$ = 3.33 cc/ min (Carry to three places.)

Step 3: 3.33 · 15 = 49.95 or 50 gtts./ min.
(Round answer to nearest drop)

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EXERCISES

1. An IV of 1500 c.c. is to infuse over 12 hours. The df of the set is 15. How many drops (gtts.) per minute should the IV run?

ANS: 42 gtts/min

2. You are to give 3000 ml. D-5-RL intravenously over the next 36 hours. For how many drops per minute should you set the drip rate? (df=15)

ANS. 21 gtts/min

3. You are to administer an intravenous infusion of 240 ml. to a child over a period of ten hours. Using a pediatric administration set (microdrip) the infusion should be set to run at how many drops a minute? (df 60)

ANS: 24 gtts/min

4. You are to give 500 c.c. D-5-W over an 8 hour period. How many c.c.'s will be left in the bottle at the end of 2 hours? (df 15)

ANS: 375 cc

OPTICAL TECHNOLOGY

Math problems utilized in Optical Technology are, in general, not as involved as those in other technical disciplines. Basically, fundamentals of algebra, geometry, and trigonometry are necessary. The problems in the next section are to provide the student with an introduction to a basic type mathematics encountered in optical technology.

The first concept discussed involves a function known as DECENTRATION. This procedure determines the proper placement of a physical component in the lens, keeping a horizontal frame of reference.

The physical component is called OPTICAL CENTER: the one point on a lens where light enters the lens front and travels through, without deviation, out the lens back.

Decentration

When determining the decentration, all measurements are in millimeters (mm) and consist of the following:

1) P. D.

The distance between the pupils when patient is looking straight ahead.

2) Boxing System

Drawing a box around the lens shape in order to determine maximum width and height

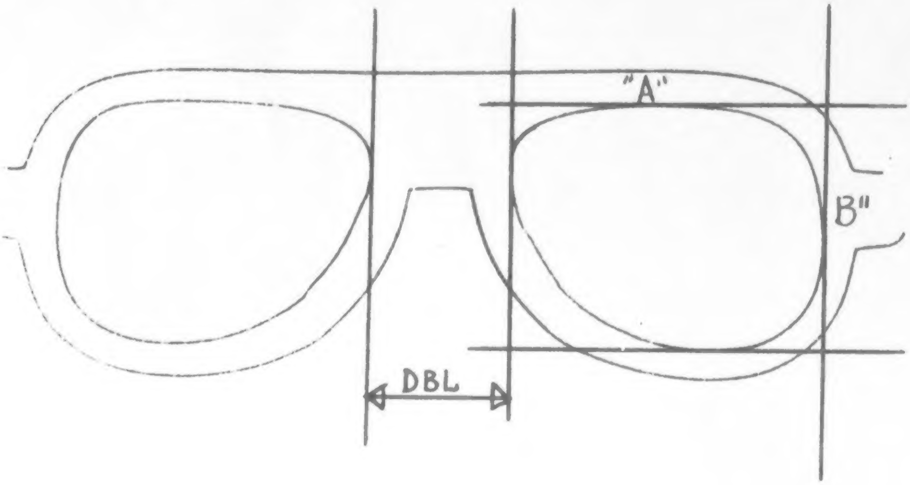
3) "A" Measurement

The horizontal measurement found after employing the boxing system.

4) "B" Measurement

Length of the "box" vertical lines.

When using the boxing system, lines are drawn and touching the lens outline only at the most extreme horizontal and vertical extensions of the outline. In math terms, these lines are said to be "tangent" to the frame:



All angles formed at the junctions of the tangent lines are 90 angles.

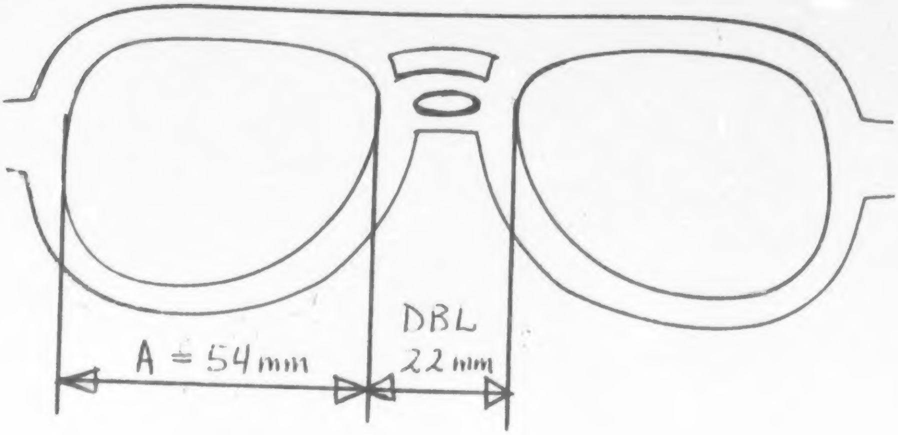
Also used in determining decentration is DBL: "the least distance that the lenses approach each other," sometimes referred to as 'the bridge.' And the FPD, calculated by:

FPD → Frame P.D. = "A" + DBL

Decentration Problem:

Given: PD = 66 mm
"A" = 54 mm
DBL = 22 mm

How much should each lens be decentered "in" (toward the nose) for the patient's pupils to coincide with the optical centers of the lens?



Solution:

1) Find the Frame PD:

FPD = "A" + DBL

FPD = 54mm + 22 mm

∴ FPD = 76 mm

2) Find the Total Decentration* for doing entire job:

Total Decentration = FPD - PD

TD = 76 mm - 66 mm

TD = 10 mm

3) Find the Decentration for each eye:

$$\text{Decentration} = \frac{TD}{2}$$

$$\text{Dec} = \frac{10 \text{ mm}}{2}$$

Dec = 5 mm for each eye

*We speak of total decentration when referring to a pair of glasses. Decentration is for each eye.

Note: Decentration is considered "in", meaning "toward the nose", if the patient's P.D. is less than the Frame PD.

Expressed mathematically:

$$\text{"in"} \rightarrow PD < FPD$$

If the patient's PD is greater than the Frame PD, decentration is considered "out".

Expressed mathematically:

"out" → PD > FPD

Further Problems

Construct a diagram for every problem and calculate total decentration with decentration and direction (i.e., "in" or "out") for each eye.

I. Given: DBL = 20 mm
"A" = 50 mm
PD = 62 mm

II. Given DBL = 14 mm
"A" = 44 mm
PD = 62 mm

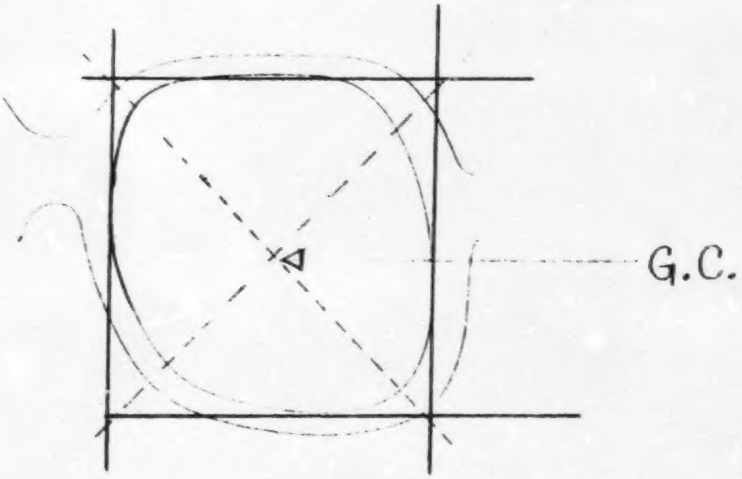
III. Given DBL = 22 mm
"A" = 44 min
PD = 66 mm

Segment Height

Segment Height refers to the vertical placement of the bifocal or trifocal segment in the frame. The decision as to what will be the segment height is arrived at by the use of measurement devices inserted into the patients frame during the analysis stage of determining prescription parameters. Segment is also known by the abbrevation "seg".

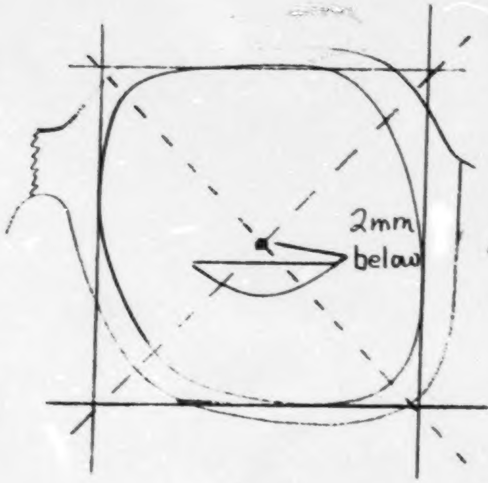
Segment height can be referred to by one of two ways:

1. "Below" or "Above" measurement in relation to the geometrical center (GC) of the lens.

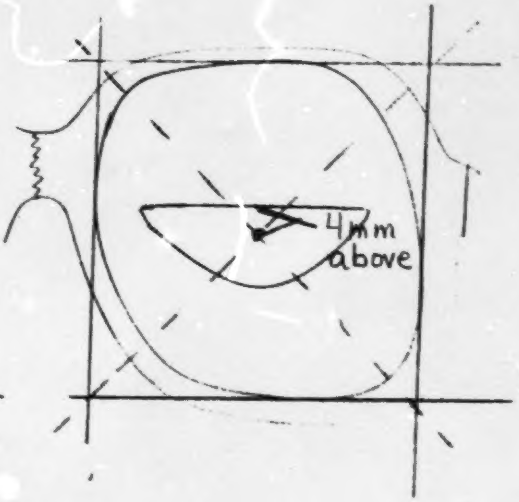


The GC is located by "boxing" a lens and drawing diagonals across the lens. The point at where the diagonals cross is the Geometrical Center.

"Above" or "Below" refers to the measured distance, in millimeters (above or below the geometrical center) to where the top of the segment will fall:



BIFOCAL TYPE

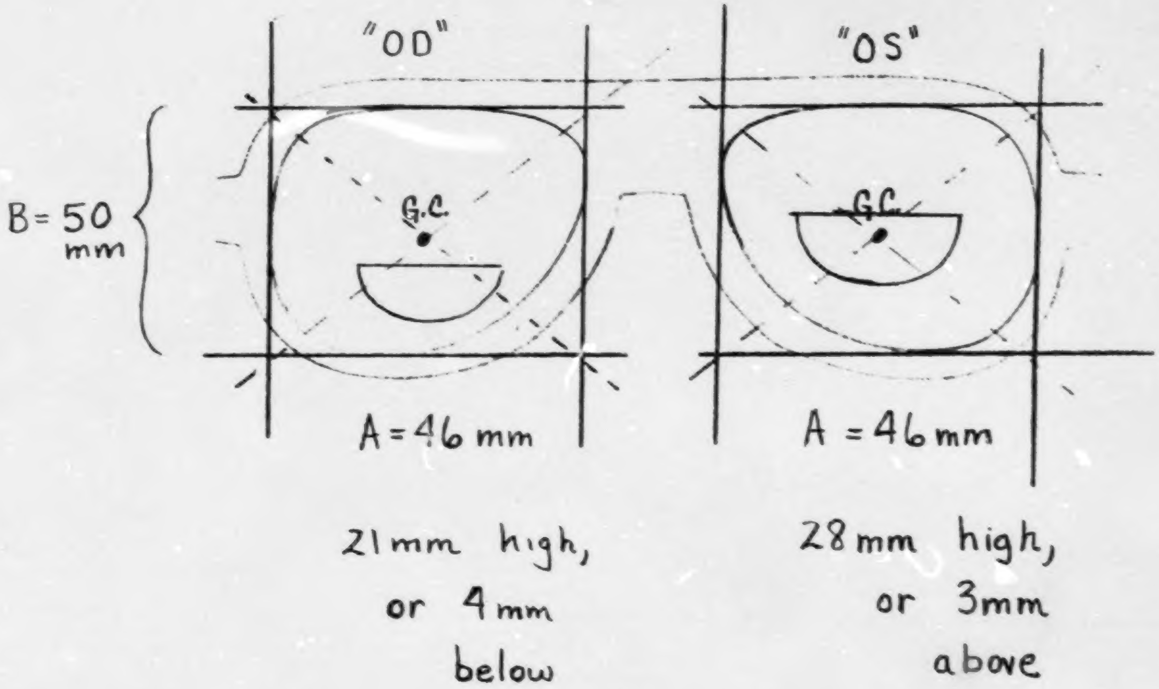


TRIFOCAL TYPE

The other method of referring to segment height is:

2. Measurement in relation to the "B" measurement of the frame, normally termed "high".

"High" refers to the distance from the lowest extension of the lens "box" to the top of the bifocal. During the manufacturing process, the "high" measurement must be transformed into one of "above" or "below":


"OD"
"OS"
B = 50 mm
G.C.
G.C.
A = 46 mm
A = 46 mm
21mm high, or 4mm below
28mm high, or 3mm above

GC Vertical Location is 25 mm.

Found mathematically,

$$\text{GC Vertical Location} = \tfrac{1}{2} \cdot B$$

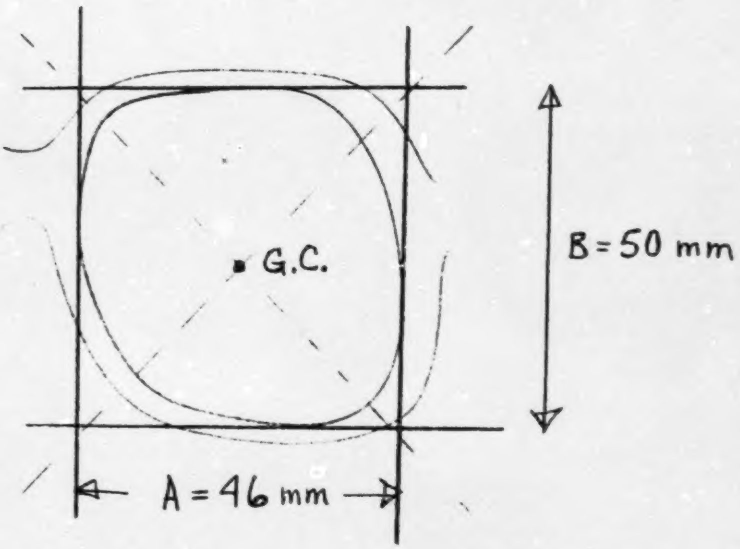
$$= \tfrac{1}{2} \cdot 50 \text{ mm}$$

$$= 25 \text{ mm}$$

EXAMPLE:

If lens OD needs a segment placed 21 mm high, what will be the measured distance from the G.C.? And, will the distance be classified as "above or "below"?

LENS "OD"



Observations:

1. The top of the segment will be 21 mm high.

2. G.C. = (½)(B) = (½)(50 mm) = 25 mm

3. Distance segment tip will be located from G.C.

Measured Distance = GC - High Measurement*
= 25 mm - 21 mm
= 4 mm

Note:

*The high measurement was less than the GC measurement. Therefore segment top will be located BELOW GC and is termed, "Below".

Final Answer: 4 mm below

Problems

1. Where would you set a segment if the "B" measurement = 50 mm and the segment was ordered 21 mm high? Determine either above or below and number of mm.

2. If the "B" measurement = 39 mm, and the segment ordered was 21 mm high, where is the bifocal going to be set vertically? Above or Below? Number of mm?

If lens "OS" needs a segment placed 28 mm high, what will be the measured distance from the GC and will it be termed "above" or "below"?

G C = 25 mm high

Ordered Segment Height = 28 mm high

Measured Distance = Ordered Height (-) G C
(The ordered height is greater than G C)

Measured Distance = 28 mm - 25 mm

Measured Distance = 3 mm

Since the ordered height is > G C, the location designation will be "above".

Answer = 3 Above

Note: Both of the measured distances were positive values. The direction of location (Above or Below) placed them in proper perspective to the geometrical center of the lens.

Problems:

1. A segment is to be placed 4 mm above; the "B" measurement of the frame is 48 mm. What was the original segment height ordered in terms of "high"?

2. A segment will be placed 26 mm high; "B" measurement = 50 mm. Where will the segment be located in respect to the G.C.?

Prism

How light moves through and out of a prism is the controlling factor in all lens systems.

Plus or converging lenses can be termed as an infinite collection of prisms with their bases at the center of the lens.

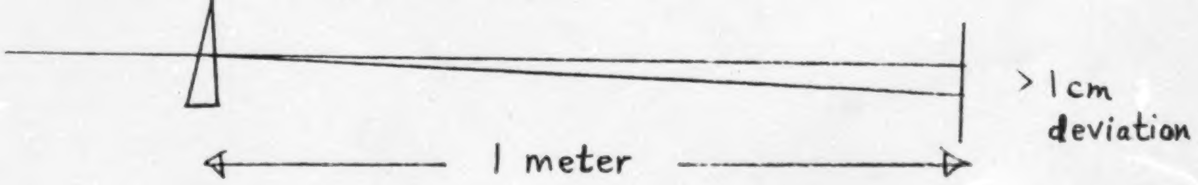
Minus or diverging lenses can be termed as an infinite collection of prisms with their bases at the edge of the lens.

All of our lens concepts are based on the expansion of the knowledge gained through the study of the prism. The unit of measurement of the deviation power of a prism is the "Prism Diopter Deviation", (noted as the symbol "Δ" after a value). The unit was conceived and named by C. F. Prentice in 1890. It has since become known as Prentice's Rule.

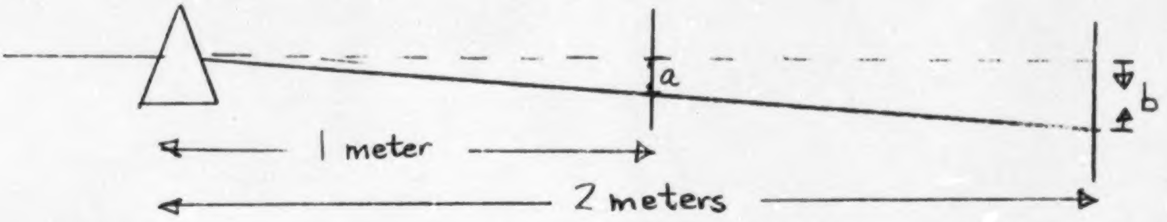
> Prentice's Rule = A 1 prism diopter prism deviates a ray of light 1 centimeter in one meter.

For example: If you have a 2 prism diopter prism, the deviation of light will be 2 cm in a travelled distance of 1 meter; a deviation of 4 cm in a travelled distance of 2 meters.

1 Prism Diopter Prism


> 1 cm
deviation
1 meter

2 Prism Diopter Prism


a
b
1 meter
2 meters

a = 2 cm deviation

b = 4 cm deviation

What would be the measured deviation of a ray of light through a 3 prism diopter prism if the measurement were taken at a distance of 5 meters from the prism?

Prism = 3 Prism Diopters

Measured Distance = 5 meters

For each prism diopter of power a ray of light is deviated 1 cm for each meter travelled. Therefore;

The length of travel = 5 meters
Strength of prism = 3 prism diopters

(Distance Travelled) •	(Strength of Prism)	=	Amount of deviation in cm.

An example:

(5 meters) • (3 Prism Diopters of Power) = 15 cm deviation at a distance of 5 meters.

A ray of light is deviated 30 mm in a distance of 2 meters. What is the power of the prism?

Solution:

I. First, convert the deviation into cm, the units of measurement used in the prism diopter deviation relationship.

30 mm = 3 cm, therefore
deviation = 3 cm.

II. Note the travelled distance = 2 meters.

We need to know what happens in 1 meter because that fact will give us the power of the prism:

588

$$\frac{\text{what we know}}{\text{what we have}} = \frac{1 \text{ meter}}{2 \text{ meters}} = 0.5 \text{ (the factor for 1 meter)}$$

The value of deviation at 1 meter = (0.5 · 3 cm),

$$\begin{array}{rl} 3 \text{ cm}, & \text{deviation at full distance} \\ \underline{\times\ .5}, & \text{the factor for distance} \\ 1.5 \text{ cm} & \end{array}$$

Therefore, deviation at 1 meter = 1.5 cm.

And, if 1 cm deviation in 1 meter is our base formula, then the power of the prism is 1.5 prism diopters.

Problems

1. If Δ = 2, and measured distance = 3 meters, what is the amount of deviation at 3 meters?

2. Given deviation = 10 cm, and measured distance = 5 meters; what is the power of Δ?

Transposition

Power components in a prescription can be:

A. Sphere: A power component which is in effect all over the lens. (May be the only component in a lens).

B. Cylinder: A power component which is "localized", meaning that in one area, the full effect of the component is evident while 90° away there is no trace of the component. (Further discussion in optical cross-section).

C. Axis: Tells the location of the cylinders. By definition there is no power of the cylinder at its axis.

Prescriptions can be written with only a sphere power.

+ 2.00 Sphere   or   - 2.00 Sphere

Or they can be written in the form known as a sphere - cylinder (a sphere with a cylinder component).

+ 2.00 + 1.00 x 90°
↓ ↓
(sphere)+(cylinder)x(axis of cylinder)

or

- 2.00 - 1.00 x 90°

Notice the 2 sphero-cylinder prescriptions given as examples. Look at the cylinder components; one is (+) while the other is (-). There is a significance to this occurrence.

As a refractionist (the person who evaluates your vision needs) is deciding what lens power is needed to correct your vision, he is doing the refraction in one of two ways. The designation of the type of refraction is by the form that the cylinder power will take.

If the refraction is done in plus cylinders, the prescription will have, if it is needed, a plus (+) cylinder component, and vice versa. By using a conversion method called "Transposition", it can be noted that whether the refraction is done in plus cylinders, or minus cylinders, the net result is the same for the patient.

Procedure for Transposition

Using an example, the rules for transposition will be listed:

Given: + 3.00 - 2.00 x 90°

I. Algebraically add the cylinder component to the sphere component:

$$\begin{array}{l} +3.00 - 2.00 \times 90^{\circ} \\ \underline{-2.00\qquad\qquad\quad} \\ +1.00 \end{array}$$

II. Change the sign of the original cylinder component:

$$\begin{array}{l} +3.00 - 2.00 \times 90^{\circ} \\ \underline{-2.00\qquad\qquad\quad} \\ +1.00 + 2.00 \times 90^{\circ} \end{array}$$

III. Rotate the axis component 90°:

Note, the resultant axis cannot be greater than 180°, or less than 0°.

$$\begin{array}{l} +3.00 \quad - \quad 2.00 \quad \text{x} \quad 90^{\circ} \\ \underline{-\ 2.00\ \text{(change sign)(rotate)}} \\ +1.00 \quad + \quad 2.00 \quad \text{x}\ 180^{\circ} \end{array}$$

Therefore, transposition is indicated:

$$+3.00 - 2.00 \text{ x } 90^{\circ} = +1.00 + 2.00 \text{ x } 180^{\circ}$$

original transposition

The concept of an axis of 0° is never used in optical dispensing. An examination of a protractor will reveal that the 0° line and the 180° line, coincide. Therefore, axis 0° is always referred to as axis 180°.

To further clarify: We can speak of "power" being localized in a lens if we restrict our viewing to one particular part of a lens.

The concept of meridians is used to speak of "localization of power". A meridian of a power is considered to go from one side of the lens to the other, crossing the optical center (point of no power in a lens). Power along a meridian is considered to be constant. As a meridian goes from one side of the lens to the other, we only have to deal with a 180° system.

If you speak of power along axis 10 (meridian 10), are you not also speaking of power along meridian 190?

Problems:

A) Transpose + 3.00 + 0.75 x 021° *

B) Transpose - 2.00 - 1.25 x 135°

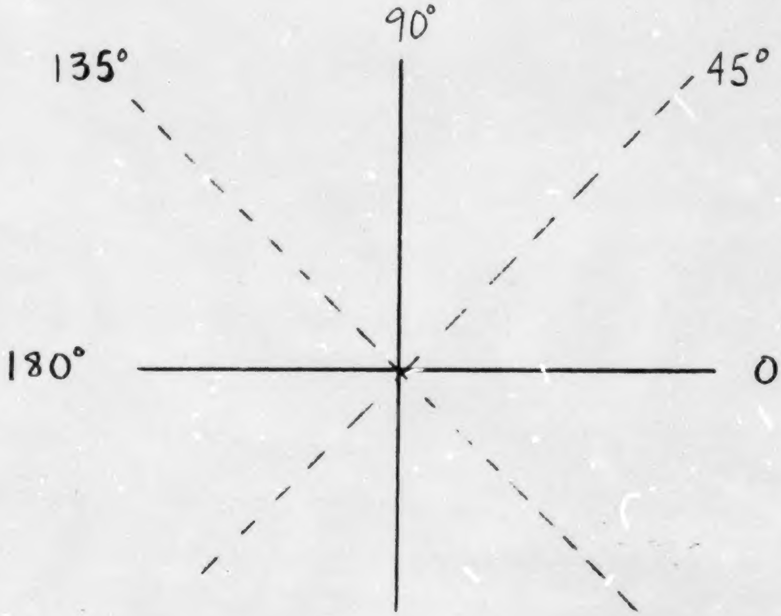
C) Transpose + 2.00 - 3.00 x 012°

* In **optics**, a 21° measurement can be expressed as 021°.

Optical Cross

The optical cross is a graphic representation of the "**effective** power" in a lens. It will show you what powers have to be built into a lens, and where, to satisfy the prescription components ordered by a refractionist.

The Optical Cross



Rules governing optical cross are;

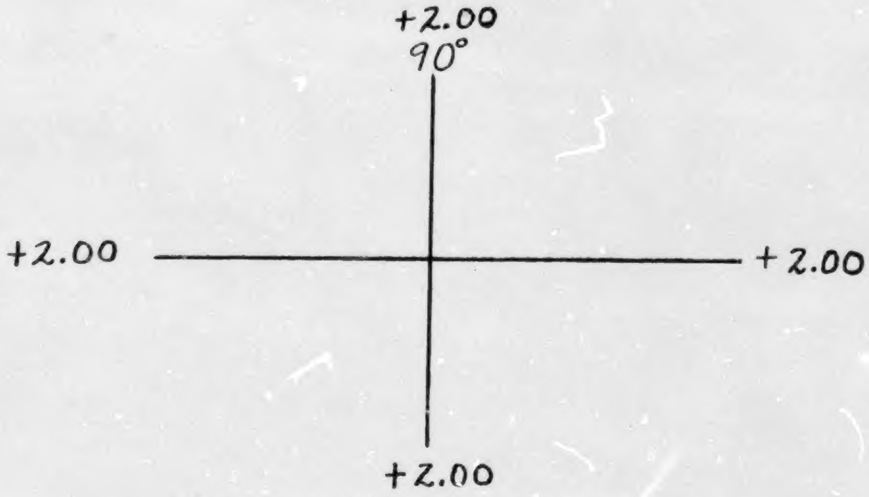
I. The sphere power is in full effect all over the lens (in all meridians, the power is constant).

II. The cylinder is in "full effect", 90° away from it's given axis (the axis in the prescription) (At the given axis there is no power of the cylinder component.)

III. Addition of the factors plotted on the optical cross will give you the effective powers in the main meridians.

EXAMPLE 1:

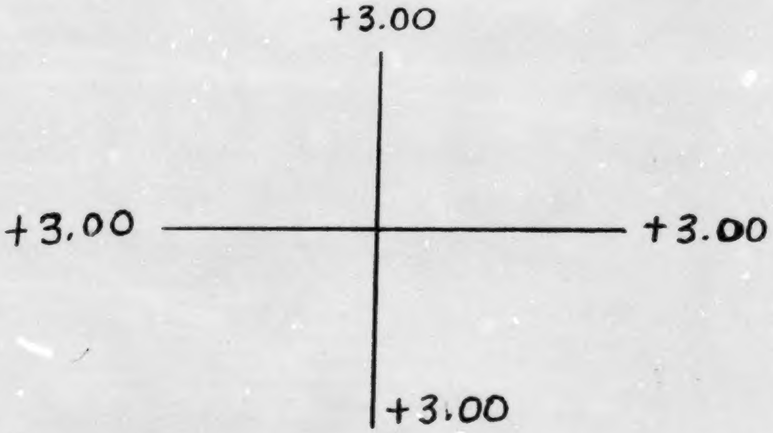
Rx = + 2.00 sphere


+2.00
90°
+2.00
+2.00
+2.00

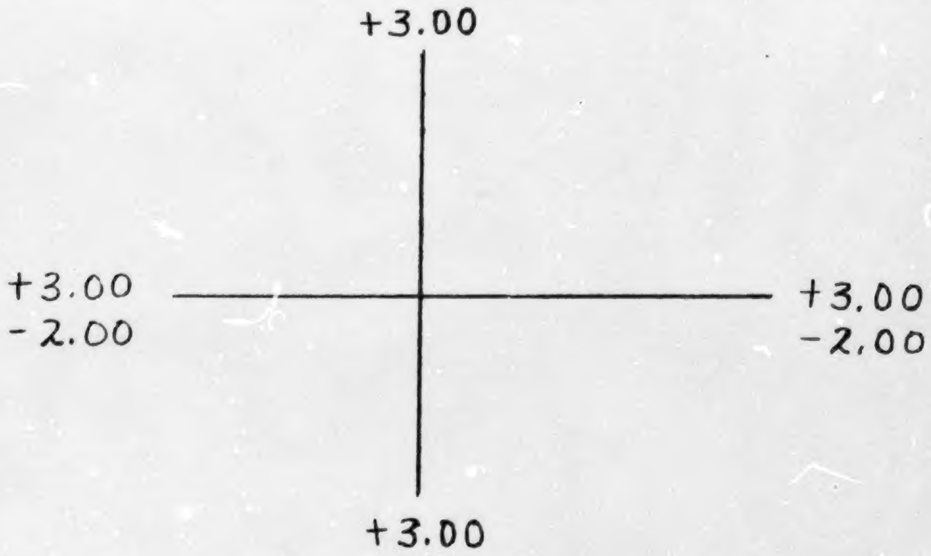
EXAMPLE 2:

Rx = + 3.00 - 2.00 x 090 (Rx from the transposition problem).

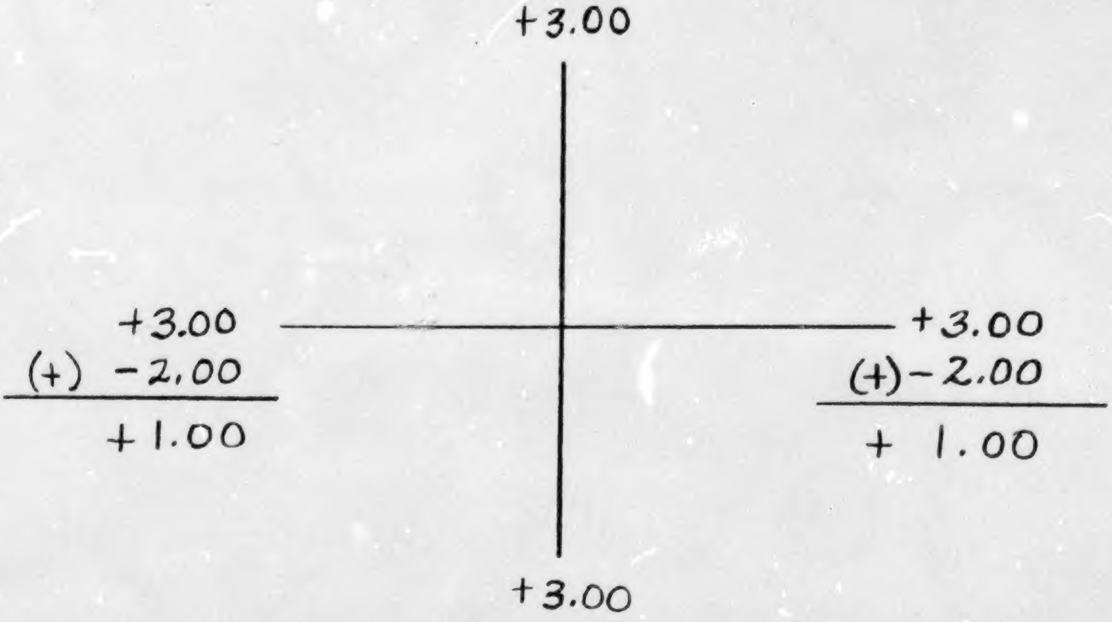
Rule 1. Sphere is in effect all over the lens.


+3.00
+3.00
+3.00
+3.00

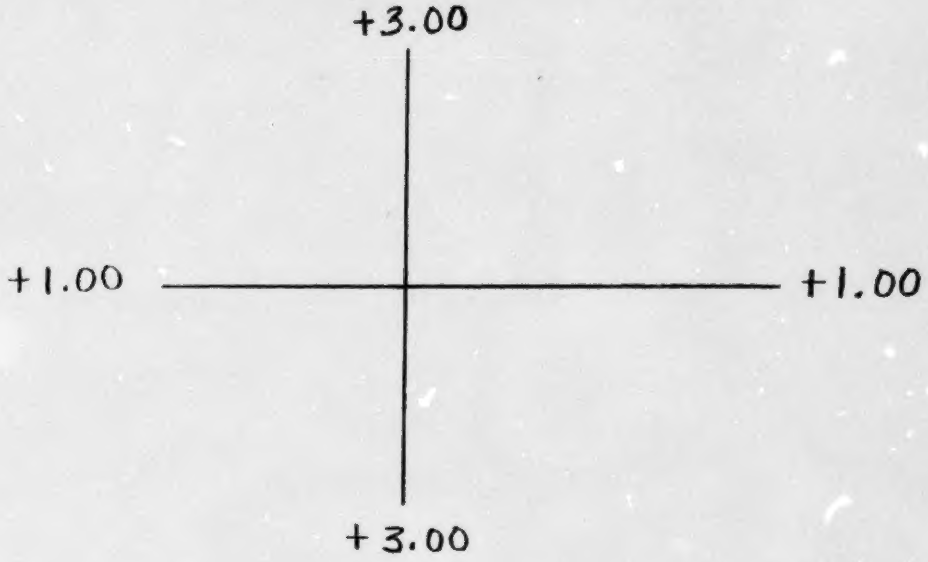
Rule 2. The cylinder is in full effect 90° away from the given axis.


+3.00
+3.00
-2.00
+3.00
-2.00
+3.00

Rule 3. Addition of powers in the key meridians will give you the effective power of the lens.


+3.00
+3.00
(+) -2.00
+1.00
+3.00
(+) -2.00
+ 1.00
+3.00

The effective power of the lens is as follows:



Along the 180 meridian, the effective power is + 1.00, while along the 90th meridian the effective power is + 3.00.

Remember our transposition problem? It concluded with this relationship:

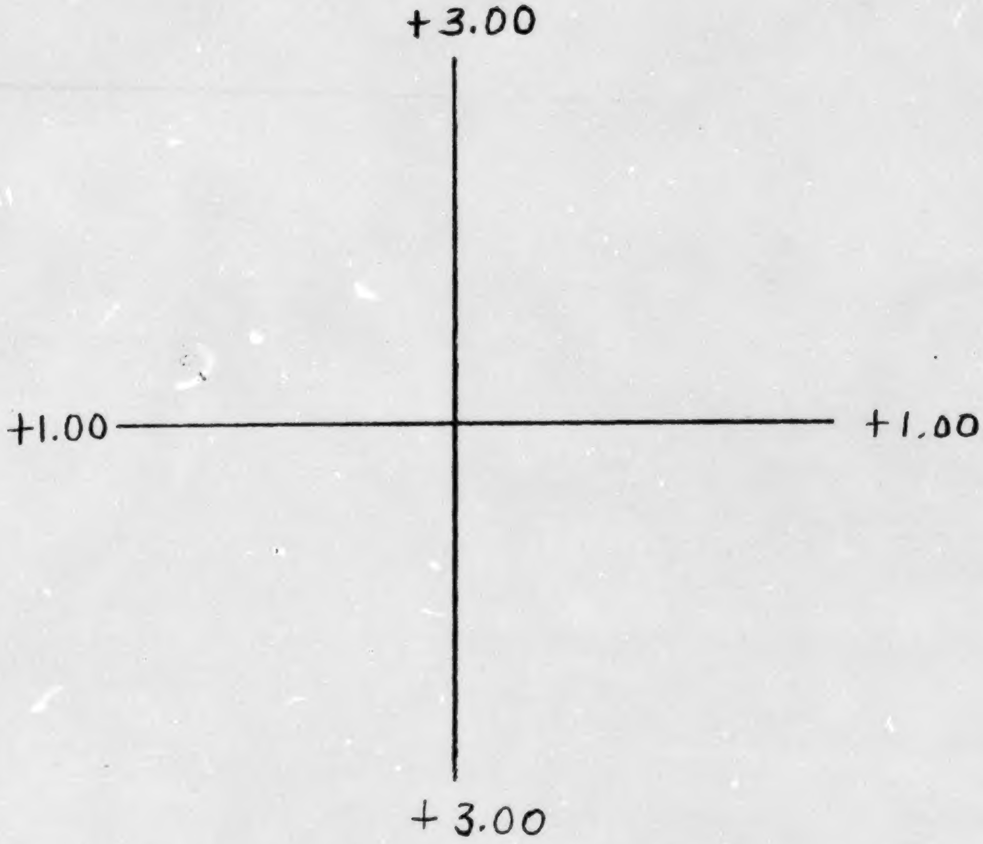
$$+ 3.00 - 2.00 \times 090 = + 1.00 + 2.00 \times 180$$

By the use of the optical cross, we can prove this relationship.

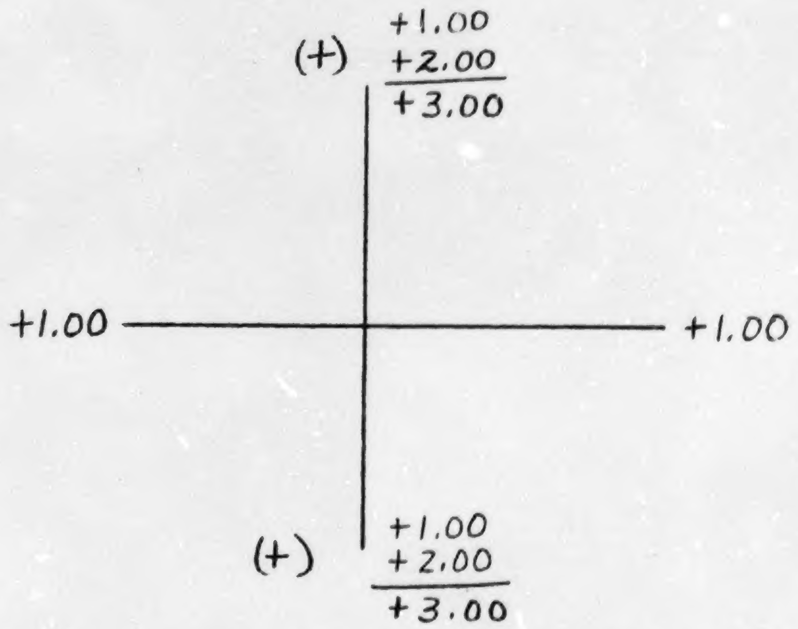
Proof of Transposition Using Optical Cross

From our optical cross example:

+ 3.00 - 2.00 x 090° = Effective power of + 3.00
along the 90
+ 1.00 along the 180



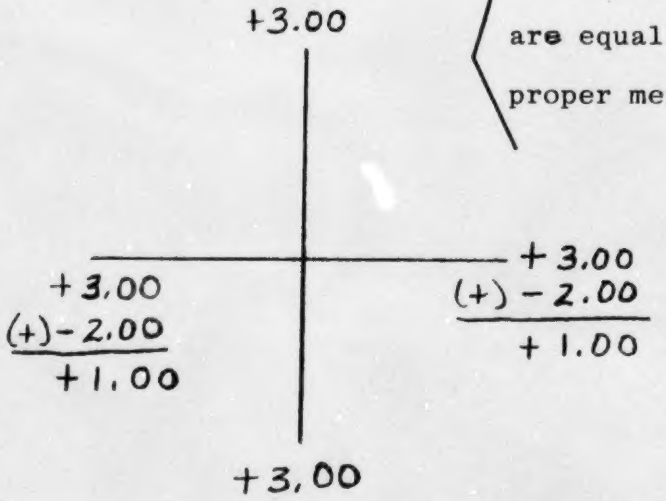
Examine the relationship of + 1.00 + 2.00 x 180:

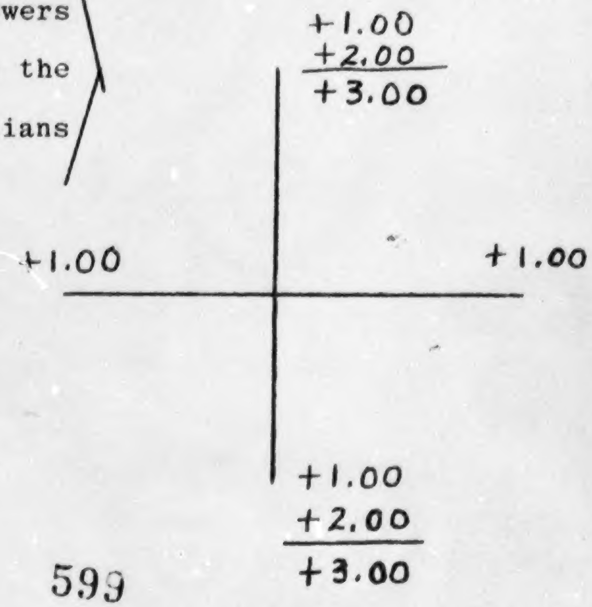

(+)
+1.00
+2.00
+3.00
+1.00
+1.00
(+)
+1.00
+2.00
+3.00

Effective power is what we are interested in.

+ 3.00 - 2.00 x 090° = + 1.00 + 2.00 x 180°

Effective powers are equal in the proper meridians


+3.00
+3.00
(+)-2.00
+1.00
+3.00
(+) - 2.00
+ 1.00
+3.00


+1.00
+2.00
+3.00
+1.00
+1.00
+1.00
+2.00
+3.00

Problems (Optical Cross)

1. Place + 2.00 + 1.75 x 45° on the optical cross.

    (a) What is the effective power along the 45th meridian?

    (b) What is the effective power along the 135th meridian?

2. Place + 2.00 - 4.00 x 90° on the optical cross.

    (a) What is the effective power along the 90th meridian?

    (b) What is the effective power along the 180th meridian?

Power of the Cylinder From It's Axis To a Point 90° Away

The power of a cylinder component varies between 0% and 100%. To find the power of a cylinder at any meridian, a little trigonometry and logic will help.

Given the formula:

$$F' = (F)(\sin^2 \theta), \text{ where}$$

F' = Power of a cylinder at any meridian

F = Total power of the original cylinder

θ = The angle between the axis of the original cylinder and the meridian in question.

Example:

Given: The prescription → Rx = + 4.00 x 090

Note: A prescription in this form, (cylinder power) • (axis) only, indicates the (sphere) is understood to be zero.

Find: What is the effective power of the cylinder at axis 45°?

Solution:

$$\text{Use } F' = (F)(\sin^2 \theta), \text{ where in this case:}$$

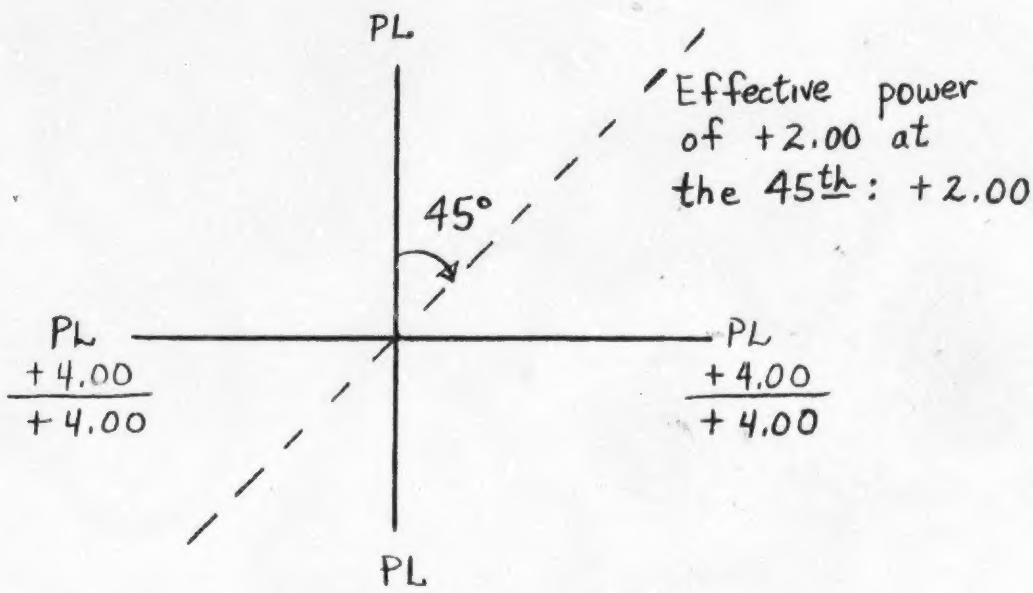
$$F = +4.00 \text{ and } \theta = 90^\circ - 45^\circ = 45^\circ$$

$$\sin 45 = 0.707 \quad \text{and } (\sin 45\ )^2 = 0.5$$

Therefore:

$$F \quad F' = (+4.00)(0.5) \qquad F' = +2.00$$

The lens in question, + 4.00 x 90°, has an effective power of + 2.00 at the 45th meridian.


PL
Effective power of +2.00 at the 45th: +2.00
45°
PL
+4.00
+4.00
PL
+4.00
+4.00
PL

Consider if there is a sphere power involved. Given,

Rx = + 2.00 + 5.00 x 45°

Once again, the effective power in any meridian will vary as to the axis of the cylinder and the degrees away from that axis of the meridian in question. Therefore, if:

+ 2.00 + 5.00 x 45°

What is the effective power of the lens along the 90th meridian? (θ will not = 90°, θ will = 45°, because 90° is 45° away from the 45th meridian).

Formula $F' = F \text{ sphere} + F \text{ cylinder } (\sin^2 \theta)$

Solution:

$$F' = F_{\text{sphere}} + (F_{\text{cylinder}} (\sin^2 \theta)), \text{ where}$$

$$F_{\text{sphere}} = +2.00$$

$$F_{\text{cyl}} = +5.00$$

Axis of Cylinder = 45

$$\theta = 90^\circ - 45^\circ = 45^\circ$$

Therefore,

$$F' = (+2.00) + ((+5.00)(\sin 45)^2)$$

$$F' = 4.50$$

Problems:

Given: Rx = + 3.00 - 4.00 x 180

What is the effective power of the lens at axis:

(a) 90°

(b) 45°

(c) 30°

(d) 60°

(e) 180°

Remember values "a through e" are not equal to "θ". You must use these values to arrive at a value which is equal to "θ".

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